(a) For The Scalar Function V = 2xy - Xz, Determine Its Directional Derivative Along The Direction Of

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When working with multivariable calculus, one of the key concepts is understanding how a scalar function changes as we move in a specific direction at a point in space. The directional derivative provides this information, measuring the rate at which the function V = 2xy - xz changes as you move from a particular point in a specified direction. In this article, we will explore how to determine the directional derivative of the scalar function V = 2xy - xz along a given direction, breaking down the process step-by-step, and highlighting important concepts involved in the calculation.

Understanding the Scalar Function V = 2xy - xz

Before diving into the calculation of the directional derivative, it is important to understand the nature of the function V = 2xy - xz. This is a scalar function of three variables, x, y, and z, which can represent a wide range of physical or mathematical phenomena.

Function Components

- The term 2xy indicates the interaction between variables x and y, scaled by a factor of 2.

- The term -xz demonstrates a relationship between variables x and z, with a negative sign indicating an inverse relationship.

Graphical Interpretation

While visualizing this three-variable function is complex, understanding its gradient and how it changes in space provides insight into its behavior, especially when analyzing directional derivatives.

Fundamentals of the Directional Derivative

The directional derivative of a scalar function V at a point P in the direction of a unit vector u (which indicates the direction of movement) is denoted as DuV and calculated as:

$$ D_u V = \nabla V \cdot \mathbf{u} $$

where:

- ∇V is the gradient vector of V,
- $\mathbf{u}$ is the unit vector indicating the direction,
- $\cdot$ signifies the dot product.

This formula emphasizes that the rate of change of the scalar function in a particular direction depends on both the gradient of the function and the chosen direction.

The Gradient Vector ∇V

The gradient vector combines all partial derivatives of V:

$$\nabla V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right)$$

Calculating this vector is the first step in determining the directional derivative.

Unit Vector $\mathbf{u}$

The direction vector must be a unit vector. If the given direction vector is not of unit length, normalize it by dividing it by its magnitude:

$$\mathbf{u} = \frac{\mathbf{d}}{\|\mathbf{d}\|}$$

where $\mathbf{d}$ is the original direction vector.

Calculating the Gradient of V = 2xy - xz

Let's compute the partial derivatives of V with respect to x, y, and z.

Partial derivative with respect to x

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x} (2xy - xz) = 2y - z$$

Partial derivative with respect to y

$$
\frac{\partial V}{\partial y} = \frac{\partial}{\partial y} (2xy - xz) = 2x
$$

Partial derivative with respect to z

$$
\frac{\partial V}{\partial z} = \frac{\partial}{\partial z} (2xy - xz) = -x
$$

Gradient vector ∇V

Putting these together:

$$
\nabla V = (2y - z, 2x, -x)
$$

This vector points in the direction of the steepest ascent of V at a given point.

Determining the Direction Vector $\mathbf{u}$

Suppose we are given a specific direction vector $\mathbf{d} = (d_x, d_y, d_z)$. To find the directional derivative, normalize this vector:

$$

$$
\mathbf{u} = \frac{(d_x, d_y, d_z)}{\sqrt{d_x^2 + d_y^2 + d_z^2}}
$$

For example, if the direction vector is $\mathbf{d} = (1, 2, 3)$, then:

$$
\|\mathbf{d}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}
$$

Thus:

$$
\mathbf{u} = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right)
$$

Calculating the Directional Derivative

With the gradient vector and the unit direction vector established, the next step is to compute the dot product.

Step 1: Plug in the point of interest

To evaluate the directional derivative at a specific point (x_0, y_0, z_0), substitute these coordinates into ∇V:

$$
\nabla V|_{(x_0, y_0, z_0)} = (2 y_0 - z_0, 2 x_0, -x_0)
$$

Step 2: Calculate the dot product

Compute:

$$
D_u V = \nabla V \cdot \mathbf{u} = (2 y_0 - z_0) \cdot u_x + 2 x_0 \cdot u_y + (-x_0) \cdot u_z
$$

where u_x, u_y, u_z are components of the unit vector $\mathbf{u}$.

Step 3: Final expression

Putting it all together, the directional derivative at the point is:

$$
D_u V = (2 y_0 - z_0) \frac{d_x}{\|\mathbf{d}\|} + 2 x_0 \frac{d_y}{\|\mathbf{d}\|} - x_0 \frac{d_z}{\|\mathbf{d}\|}
$$

This value provides the rate at which the function V changes as you move from (x_0, y_0, z_0) in the specified direction.

Practical Applications and Significance

Understanding how to compute the directional derivative of functions like V = 2xy - xz has a wide range of applications:

- **Optimization:** Finding maximum or minimum values of functions in specific directions.
- **Gradient-based algorithms:** Used in machine learning and numerical methods to improve convergence.
- **Physical interpretations:** In physics and engineering, the directional derivative can represent the rate of change of a physical quantity (like temperature or pressure) along a certain direction.
- **Surface analysis:** Analyzing how a surface defined by the scalar function varies in space.

Summary

Calculating the directional derivative of the scalar function $V = 2xy - xz$ involves several key steps:

1. Compute the gradient vector ∇V by finding the partial derivatives with respect to x, y, and z.
2. Normalize the given direction vector to obtain a unit vector $\mathbf{u}$.
3. Calculate the dot product of ∇V with $\mathbf{u}$ at the point of interest.
4. Interpret the resulting value as the rate of change of V in that specific direction.

Mastering this process enables a deeper understanding of multivariable functions and their behaviors across different directions in space, which is essential for advanced studies in calculus, physics, and engineering.

Keywords: directional derivative, scalar function, gradient vector, partial derivatives, vector calculus, multivariable calculus, optimization, physics applications, surface analysis

Frequently Asked Questions

What is the scalar function V = 2xy - xz, and how is its directional derivative defined?

The scalar function V = 2xy - xz represents a multivariable function, and its directional derivative at a point measures the rate of change of V in a specified direction. It is computed as the dot product of the gradient of V and a unit vector in the given direction.

How do you compute the gradient of the scalar function V = 2xy - xz?

The gradient ∇V is computed by taking partial derivatives with respect to each variable: $\partial V/\partial x = 2y - z$, $\partial V/\partial y = 2x$, and $\partial V/\partial z = -x$. Thus, $\nabla V = (2y - z, 2x, -x)$.

What is the general formula for the directional derivative of V in a given direction?

The directional derivative of V at a point in the direction of a unit vector $u = (u_x, u_y, u_z)$ is given by $D_u V = \nabla V \cdot u = (\partial V/\partial x)u_x + (\partial V/\partial y)u_y + (\partial V/\partial z)u_z$.

How do you find the unit vector in the specified direction for the directional derivative?

The unit vector u is obtained by dividing the given direction vector by its magnitude: $u = v / ||v||$, where v is the direction vector and $||v||$ is its length.

If the direction is along the vector v = (a, b, c), how is the directional derivative calculated?

First, normalize v to get $u = (a, b, c) / \sqrt{(a^2 + b^2 + c^2)}$. Then, compute ∇V at the point of interest and take the dot product with u to find the directional derivative.

What is the significance of the gradient ∇V in determining the maximum rate of change of the scalar function?

The gradient ∇V points in the direction of the steepest ascent of V, and its magnitude gives the maximum rate of increase of V at the point. The directional derivative in the direction of ∇V is the largest possible value.

How would you evaluate the directional derivative of V = 2xy - xz along a specific direction at a given point?

Identify the point and the direction vector, normalize the direction vector to a unit vector, compute the gradient at that point, and then take the dot product of the gradient with the unit vector to find the directional derivative.

Why is it important to normalize the direction vector when calculating the directional derivative?

Normalizing ensures that the magnitude of the direction vector is 1, which makes the directional derivative represent the rate of change per unit length in that direction, providing a consistent measure.

Can the directional derivative be zero? Under what circumstances?

Yes, the directional derivative is zero in directions where the gradient of V is orthogonal to the direction vector, indicating no change in V along that direction at the point.

Additional Resources

Scalar Function V = 2xy - xz: An In-Depth Analysis of Its Directional Derivative Along a Given Direction

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Introduction

Understanding the behavior of scalar functions and their derivatives forms the backbone of multivariable calculus, with profound applications across physics, engineering, and mathematics. In particular, the concept of the directional derivative provides insight into how a function changes as we move through space in specific directions. This article delves into the detailed derivation of the directional derivative of the scalar function $V = 2xy - xz$, exploring its mathematical structure, the process of calculation, and the geometric interpretations involved. The goal is to provide a comprehensive, accessible, yet rigorous overview suitable for students, educators, and professionals seeking to deepen their understanding of multivariable calculus.

The Scalar Function $V = 2xy - xz$: Fundamental Concepts

Definition and Significance

The function $V(x, y, z) = 2xy - xz$ is a scalar-valued function of three variables. Such functions are ubiquitous in physics for representing potential fields, in fluid dynamics for describing flow quantities, and in optimization problems. The form of V combines products of variables, hinting at its potential for interesting directional behavior, especially when considering how it varies with respect to changes in x, y, and z.

Basic Properties

- Domain: Typically, the domain of V is $\mathbb{R}^3$, with possible restrictions depending on context.
- Continuity: V is continuous everywhere in $\mathbb{R}^3$, being a polynomial.
- Differentiability: V is infinitely differentiable (smooth), allowing us to compute its gradient and directional derivatives at any point.

Understanding the Gradient and Its Role

Gradient of V

The gradient vector ∇V encapsulates the direction of greatest increase of the function at a point and is fundamental in calculating the directional derivative.

Calculation of the Gradient:

$$
\nabla V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right)
$$

- Partial derivative w.r.t. x:

$$
\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}(2xy - xz) = 2y - z
$$

$$

- Partial derivative w.r.t. y:

$$
\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(2xy - xz) = 2x
$$

- Partial derivative w.r.t. z:

$$
\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}(2xy - xz) = -x
$$

Thus,

$$
\nabla V (x, y, z) = (2y - z,\, 2x,\, -x)
$$

Geometric Interpretation

The gradient vector at any point indicates the direction of steepest ascent, with its magnitude corresponding to the rate of maximum increase. Conversely, the negative of the gradient indicates the direction of steepest descent. Understanding the gradient's structure helps contextualize the behavior of the scalar function in three-dimensional space.

The Concept of Directional Derivative

Definition

The directional derivative of a scalar function V at a point $\mathbf{r}_0 = (x_0, y_0, z_0)$ in the direction of a unit vector $\mathbf{u} = (u_x, u_y, u_z)$ measures how V changes as we move from $\mathbf{r}_0$ infinitesimally in the direction of $\mathbf{u}$. It is mathematically expressed as:

$$
D_{\mathbf{u}} V (\mathbf{r}_0) = \lim_{h \to 0} \frac{V(\mathbf{r}_0 + h \mathbf{u}) - V(\mathbf{r}_0)}{h}
$$

Relationship with the Gradient

The key relation between the directional derivative and the gradient is:

$$
D_{\mathbf{u}} V (\mathbf{r}_0) = \nabla V (\mathbf{r}_0) \cdot \mathbf{u}
$$

where $\cdot$ denotes the dot product. This formula simplifies the computation significantly, as it leverages the known gradient and the direction vector.

Importance in Analysis

The directional derivative tells us:

- How rapidly V increases or decreases in a specific direction.
- The optimal directions for maximum and minimum change.
- The local linear approximation of V around a point.

Calculating the Directional Derivative of V

Step 1: Identify the Point of Evaluation

To proceed, specify the point $\mathbf{r}_0 = (x_0, y_0, z_0)$. For illustrative purposes, suppose:

$$
\mathbf{r}_0 = (x_0, y_0, z_0)
$$

(If a specific point is provided in the problem, substitute accordingly.)

Step 2: Specify the Direction Vector $\mathbf{u}$

The question states "along the direction of" but does not specify the vector. Typically, the direction vector $\mathbf{u}$ should be:

- Given explicitly (e.g., $\mathbf{u} = (a, b, c)$), or
- Provided as a unit vector.

Assuming the vector $\mathbf{v} = (a, b, c)$ is given, normalize it to obtain $\mathbf{u}$:

$$
\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}}
$$

This normalization ensures the directional derivative is correctly scaled.

Step 3: Compute the Gradient at $\mathbf{r}_0$

Using the formula for the gradient, evaluate each component:

$$
\nabla V (x_0, y_0, z_0) = (2 y_0 - z_0,\, 2 x_0,\, -x_0)
$$

Step 4: Calculate the Dot Product

Compute:

$$
D_{\mathbf{u}} V (\mathbf{r}_0) = \nabla V (x_0, y_0, z_0) \cdot \mathbf{u}
$$

which involves multiplying corresponding components and summing.

Explicitly:

$$
D_{\mathbf{u}} V = (2 y_0 - z_0) u_x + 2 x_0 u_y - x_0 u_z
$$

Analytical Example: Practical Computation

Suppose the point of interest is $\mathbf{r}_0 = (1, 2, 3)$, and the direction vector is $\mathbf{v} = (4, 1, 2)$.

Step 1: Normalize $\mathbf{v}$

$$
\|\mathbf{v}\| = \sqrt{4^2 + 1^2 + 2^2} = \sqrt{16 + 1 + 4} = \sqrt{21}
$$

\]

\[
\mathbf{u} = \left(\frac{4}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{2}{\sqrt{21}} \right)
\]

Step 2: Compute the Gradient at \((1, 2, 3) \)

\[
\nabla V = (2 \times 2 - 3,\, 2 \times 1,\, -1) = (4 - 3,\, 2,\, -1) = (1, 2, -1)
\]

Step 3: Calculate the Dot Product

\[
D_{\mathbf{u}} V = (1) \times \frac{4}{\sqrt{21}} + (2) \times \frac{1}{\sqrt{21}} + (-1) \times
\frac{2}{\sqrt{21}}
\]

\[
D_{\mathbf{u}} V = \frac{4}{\sqrt{21}} + \frac{2}{\sqrt{21}} - \frac{2}{\sqrt{21}} = \frac{4}{\sqrt{21}}
\]

Result:

\[
D_{\mathbf{u}} V = \frac{4}{\sqrt{21}}
\]

This positive value indicates that moving in the specified direction increases the function's value at the point \((1, 2, 3) \).

Interpretations and Applications

Geometric Insights

- Direction of maximum increase: The directional derivative is maximized when \(\mathbf{u} \) is aligned with \(\nabla V \).
- Direction of maximum decrease: The derivative is minimized when \(\mathbf{u} \) points opposite to \(\nabla

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