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(b) Show That If H And K Are Subgroups Of An Abelian Group G, Then {hk : h ∈ H, k ∈ K} Is A Subgroup Of G

Introduction to Subgroups in Abelian Groups

In the study of group theory, one fundamental concept is that of subgroups—subsets of a group that themselves satisfy the group axioms. Understanding how subgroups interact within larger groups, especially in abelian (commutative) groups, provides valuable insight into the structure and properties of these algebraic systems. A particularly interesting problem is to examine the set formed by the products of elements from two subgroups.

This article aims to rigorously demonstrate that if H and K are subgroups of an abelian group G, then the set

$$
HK = \{hk : h \in H,\, k \in K\}
$$

is itself a subgroup of G. We will explore the necessary definitions, properties, and proofs step-by-step, ensuring clarity and depth of understanding.

Understanding the Basic Concepts

What Is an Abelian Group?

An abelian group is a group G with an operation (commonly denoted as addition $+$) satisfying the following properties:

- Closure: For all $a, b \in G$, $a + b \in G$.
- Associativity: For all $a, b, c \in G$, $(a + b) + c = a + (b + c)$.
- Identity Element: There exists an element $0 \in G$ such that for all $a \in G$, $a + 0 = a$.
- Inverse Elements: For each $a \in G$, there exists an element $-a$ such that $a + (-a) = 0$.
- Commutativity: For all $a, b \in G$, $a + b = b + a$.

The commutative property is what distinguishes abelian groups from non-abelian groups.

Subgroups and Their Properties

A subgroup H of a group G is a subset $H \subseteq G$ that itself forms a group under the operation of G. The criteria for H to be a subgroup are:

- Non-emptiness: H contains at least one element (usually the identity).
- Closure: For all $h_1, h_2 \in H$, the product $h_1 h_2 \in H$.
- Inverses: For all $h \in H$, the inverse $h^{-1} \in H$.

In the context of abelian groups, the operation is commutative, which simplifies many proofs.

Forming the Product of Two Subgroups

Given two subgroups H and K of an abelian group G, the set

$$
HK = \{hk : h \in H,\, k \in K\}
$$

is called the product of H and K. The question is whether this set is itself a subgroup of G.

Key intuition: Because G is abelian, the elements commute, which will be crucial in showing that HK satisfies the subgroup criteria.

Proving that HK Is a Subgroup of G

To establish that HK is a subgroup, we need to verify the following:

1. Non-emptiness: $HK \neq \emptyset$.
2. Closure under the group operation: For any $a, b \in HK$, $a b \in HK$.
3. Existence of inverses: For any $a \in HK$, $a^{-1} \in HK$.

Let's proceed step-by-step.

Step 1: Showing HK is Non-Empty

Since H and K are subgroups, they contain the identity element e of G. Therefore,

$$
e \in H \quad \text{and} \quad e \in K,
$$

which implies

$$
e \cdot e = e \in HK.
$$

Thus, the set HK is non-empty.

Step 2: Closure of HK under the Group Operation

Take any two elements $a, b \in HK$. By definition, there exist $h_1, h_2 \in H$ and $k_1, k_2 \in K$ such that:

$$
a = h_1 k_1, \quad b = h_2 k_2.
$$

We want to show that the product $a b$ is also in HK. Compute:

$$
a b = (h_1 k_1)(h_2 k_2).
$$

Since G is abelian, the elements commute, so:

$$
a b = h_1 h_2 k_1 k_2.
$$

Now, because H and K are subgroups, they are closed under the operation:

- $h_1 h_2 \in H$,
- $k_1 k_2 \in K$.

Define:

$$
h_3 = h_1 h_2 \in H, \quad k_3 = k_1 k_2 \in K.
$$

Thus,

$$
a b = h_3 k_3,
$$

which is an element of HK. Therefore, closure holds.

Step 3: Existence of Inverses in HK

Take any $a \in HK$. Then $a = h k$ for some $h \in H$, $k \in K$.

Since H and K are subgroups, they contain inverses:

$$
h^{-1} \in H, \quad k^{-1} \in K.
$$

Compute:

$$
a^{-1} = (h k)^{-1}.
$$

In an abelian group, the inverse of a product is the product of the inverses in reverse order:

$$
a^{-1} = k^{-1} h^{-1}.
$$

But because the group is abelian, $h^{-1} k^{-1} = k^{-1} h^{-1}$. Therefore,

$$
a^{-1} = h^{-1} k^{-1} \in HK,
$$

as $h^{-1} \in H$ and $k^{-1} \in K$.

Note: The key step here is the commutativity, which allows us to write the inverse of the product as the product of inverses without concern for order.

Conclusion: HK Is a Subgroup of G

Having verified all three subgroup criteria, we conclude:

$$
HK = \{hk : h \in H,\, k \in K\}
$$
is a subgroup of G.

Summary of the proof:

- The identity element is in HK.
- Closure is guaranteed by the commutative property, allowing the product of two elements in HK to be expressed as another element in HK.
- Inverses are also in HK due to the subgroup properties and the abelian nature of G.

Additional Insights and Implications

When Is HK Equal to the Subgroup Generated by $H \cup K$?

In the context of abelian groups, the set HK is not only a subgroup but also equals the subgroup generated by $H \cup K$. This is because:

$$
\langle H \cup K \rangle = HK,
$$

where $\langle H \cup K \rangle$ denotes the smallest subgroup containing both H and K.

Special Cases and Examples

- Example 1: If H and K are subgroups of $\mathbb{Z}$ (the integers under addition), then their sum $H + K$ (which is analogous to HK in additive notation) is also a subgroup.

- Example 2: In $\mathbb{Z}_n$, the cyclic group of order n, the sum of subgroups is again a subgroup, illustrating the general principle in finite abelian groups.

Conclusion

The proof that the product HK of two subgroups H and K of an abelian group G is itself a subgroup is fundamental in group theory. The abelian property—commutativity—is essential here, simplifying the proof and ensuring that the set HK retains the subgroup properties.

Understanding this property deepens our appreciation of the structure within abelian groups and provides a foundation for exploring more complex concepts such as normal subgroups,

Frequently Asked Questions

What is the main statement regarding subgroups H and K of an abelian group G?

If H and K are subgroups of an abelian group G, then the set $\{hk : h \in H, k \in K\}$ is also a subgroup of G.

Why does the abelian property of G matter in proving that HK is a subgroup?

Because in an abelian group, elements commute (hk = kh), which ensures that the product set HK is closed under multiplication and inverses, making HK a subgroup.

How do we show that HK is closed under the group operation?

For any elements h_1k_1 and h_2k_2 in HK, their product is $(h_1k_1)(h_2k_2) = h_1h_2k_1k_2$, which stays in HK because H and K are subgroups and G is abelian, so $h_1h_2 \in H$ and $k_1k_2 \in K$.

What is the significance of the subgroups H and K being normal in G?

In an abelian group, all subgroups are normal, which simplifies the structure and ensures the set HK is a subgroup without additional conditions.

Can the set HK be equal to G? Under what conditions?

Yes, if H and K generate the entire group G (i.e., their product covers G), then HK = G.

Is HK always equal to KH in an abelian group?

Yes, because the group is abelian, so the product set HK equals KH.

How does the structure of subgroups H and K influence the subgroup HK?

Since H and K are subgroups, their elements are closed under inverses and group operation, ensuring HK inherits subgroup properties in an abelian group.

What role does commutativity play in proving HK is a subgroup?

Commutativity allows rearrangement of elements within products, ensuring closure and the existence of inverses within HK, which are essential for HK to be a subgroup.

Can the set HK be a proper subset of G? Provide an example.

Yes, HK can be a proper subset of G. For example, in Z_4, subgroups H = {0, 2} and K = {0, 2} produce HK = {0, 2}, which is a proper subset of Z_4.

Additional Resources

(b) Show That If H And K Are Subgroups Of An Abelian Group G, Then {hk : h ∈ H, k ∈ K} Is A Subgroup Of G

Introduction

In the fascinating realm of group theory, the structure and interplay of subgroups within larger groups reveal profound insights into algebraic systems. When working with abelian groups—where the operation is commutative—the relationships among subgroups become particularly elegant and accessible. One notable result is that the set formed by the products of elements from two subgroups also constitutes a subgroup itself. This article delves into this concept, demonstrating that if H and K are subgroups of an abelian group G, then the set

$$
HK = \{hk : h \in H,\, k \in K\}
$$

is indeed a subgroup of G. Throughout this exploration, we will unpack the underlying principles, provide rigorous proofs, and illustrate the significance of this property in the broader context of algebra.

Understanding the Foundations: Key Definitions and Concepts

Before diving into the proof, it's essential to establish clarity on the fundamental definitions involved.

1. Groups and Subgroups

- Group G: A set equipped with a binary operation $\cdot$ satisfying four axioms: closure, associativity, identity, and invertibility.
- Subgroup $H \subseteq G$: A subset of G that itself forms a group under the operation inherited from G. For H, this requires:
- Closure under the operation.
- Containing the identity element of G.
- Closure under inverses.

2. Abelian Group

- Abelian (Commutative) Group: A group where the operation satisfies $xy = yx$ for all $x, y \in G$.

This commutative property simplifies many subgroup interactions, particularly the product of subgroups.

3. Product of Subgroups

- For subgroups $H, K \subseteq G$, the set:

$$
HK = \{hk : h \in H,\, k \in K\}
$$

is called the product of H and K. This set contains all possible products of an element from H with an element from K.

The Core Theorem and Its Significance

Theorem: If H and K are subgroups of an abelian group G, then HK is a subgroup of G.

This theorem is fundamental because it ensures that the product of two subgroups in an abelian setting retains subgroup structure. This property is instrumental in understanding the lattice of subgroups, quotient groups, and the overall architecture of abelian groups.

Step-by-Step Proof of the Theorem

Proving that HK is a subgroup involves verifying the subgroup criteria:

1. Closure under the group operation.
2. Existence of the identity element within HK.
3. Closure under taking inverses.

Let's examine each of these in detail.

---

1. Closure of HK under the Group Operation

Claim: For any $a, b \in HK$, the product $a b$ also belongs to HK.

Proof:

- Since $a, b \in HK$, there exist $h_1, h_2 \in H$ and $k_1, k_2 \in K$ such that:

$$
a = h_1 k_1, \quad b = h_2 k_2.
$$

- Compute $a b$:

$$
a b = (h_1 k_1)(h_2 k_2).
$$

- Because G is abelian, the operation is commutative, and multiplication distributes over the elements:

$$
a b = h_1 h_2 k_1 k_2.
$$

\]

- Since H and K are subgroups, they are closed under the operation, so:

$$
h_1 h_2 \in H, \quad k_1 k_2 \in K.
$$

- Therefore, $a b = (h_1 h_2)(k_1 k_2) \in HK$.

Conclusion: The set HK is closed under the group operation.

---

2. The Identity Element is in HK

Claim: The identity element e of G belongs to HK.

Proof:

- Since H and K are subgroups, they both contain the identity element e of G.

- Consider:

$$
e_H \in H, \quad e_K \in K,
$$

where $e_H = e_K = e$.

- Now, their product:

$$
e_H e_K = e \cdot e = e,
$$

which is the identity in G.

- Thus, $e \in HK$.

Conclusion: The identity element is contained in HK.

---

3. Closure Under Inverses

Claim: For any $a \in HK$, its inverse a^{-1} also belongs to HK.

Proof:

- Let $a \in HK$, so there exist $h \in H$ and $k \in K$ such that:

$$
a = h k.
$$

- Since H and K are subgroups, they are closed under inverses:

$$
h^{-1} \in H, \quad k^{-1} \in K.
$$

- Compute a^{-1}:

$$
a^{-1} = (h k)^{-1}.
$$

- In a group, the inverse of a product is:

$$
(h k)^{-1} = k^{-1} h^{-1},
$$

but in abelian groups, the order of multiplication doesn't matter, so:

$$
k^{-1} h^{-1} = h^{-1} k^{-1}.
$$

- Since H and K are subgroups and the group is abelian, $h^{-1} \in H$ and $k^{-1} \in K$, so:

$$
a^{-1} = h^{-1} k^{-1} \in HK.
$$

Conclusion: The inverse of any element in HK is also in HK.

Finalizing the Proof: HK Is a Subgroup

By verifying the three criteria:

- Contains the identity element.
- Closed under the group operation.
- Closed under inverses.

we conclude that HK is indeed a subgroup of G.

The Role of Commutativity

A pivotal aspect of this proof is the abelian property of G. The step where the inverse of the product is simplified hinges on the commutativity:

$$
(h k)^{-1} = k^{-1} h^{-1} = h^{-1} k^{-1}.
$$

If G were non-abelian, this equality wouldn't necessarily hold, and HK might not be a subgroup. In non-abelian groups, the product of subgroups doesn't automatically form a subgroup unless additional conditions are satisfied, such as normality.

Broader Implications and Applications

Understanding that the product HK forms a subgroup when H and K are subgroups of an abelian group has profound implications:

- Subgroup Lattices: It helps to structure the subgroup lattice, mapping how subgroups combine and intersect.
- Direct and Semidirect Products: This property underpins the construction of more complex groups, including direct products, which are fundamental in decomposing groups into simpler components.
- Quotient Group Construction: In the context of normal subgroups, the product of subgroups often appears in defining quotient groups.
- Algebraic Topology and Module Theory: Similar principles apply in these areas, where subgroup interactions influence the structure of modules, rings, and other algebraic objects.

Summing Up

The proof that the set $HK = \{hk : h \in H,\, k \in K\}$ is a subgroup when H and K are subgroups of an abelian group G hinges on fundamental properties of groups—closure, identity, and inverses—and crucially, the commutative nature of G. This elegant result exemplifies how algebraic structures maintain their integrity under specific operations and paves the way for more advanced explorations into group theory and its applications across mathematics.

Final Remarks

In the rich landscape of algebra, the behavior of subgroups under various operations forms a cornerstone of understanding group structures. The fact that in abelian groups, the product of subgroups is itself a subgroup, underscores the harmonious and predictable nature of commutative algebraic systems. As we continue to explore the depths of abstract algebra, these foundational principles serve as guiding lights, illuminating the path toward more complex and intriguing mathematical phenomena.

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