) Find A 3 4 Matrix A, In Reduced Echelon Form, With Free Variable X3, Such That The General Solution

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Introduction to Matrices and Their Role in Linear Algebra

Understanding matrices is fundamental to grasping the concepts of linear algebra, which underpins many scientific and engineering disciplines. Matrices serve as compact representations of systems of linear equations, enabling efficient manipulation and analysis. Among the numerous forms matrices can take, the reduced echelon form (REF) and reduced row echelon form (RREF) are particularly important because they simplify matrices to their most elementary form, revealing key properties such as solutions to the associated systems.

In this article, we focus on constructing a 3x4 matrix A in reduced echelon form that has a specific free variable, x_3, and examine how to derive the general solution for the corresponding system. This process involves understanding the structure of matrices, the concept of free variables, and the techniques used to convert matrices into RREF to analyze solutions effectively.

Understanding the Structure of a 3x4 Matrix in RREF

What Is a 3x4 Matrix?

A 3x4 matrix A consists of 3 rows and 4 columns, typically representing a system of 3 equations with 3 variables and an augmented column for constants. The matrix looks like:

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & b_1 \\
a_{21} & a_{22} & a_{23} & b_2 \\

a_{31} & a_{32} & a_{33} & b_3 \\
\end{bmatrix}
\]

where \(a_{ij} \) are the coefficients, and \(b_i \) are the constants.

Reduced Echelon Form (REF) and Reduced Row Echelon Form (RREF)

- REF is a form where:
- All non-zero rows are above rows of all zeros.
- The leading coefficient (pivot) of each non-zero row is to the right of the leading coefficient in the row above.
- The leading coefficient in each non-zero row is 1.
- All entries below the leading 1 are zeros.

- RREF extends REF by requiring:
- All leading 1s are the only non-zero entries in their columns (i.e., zeros above and below pivots).

Achieving RREF simplifies solving the system and clearly reveals the solution set.

Designing a 3x4 Matrix with a Free Variable \(x_3 \)

Key Points for Matrix Design

To incorporate a free variable \(x_3 \), the matrix's row echelon form must be designed such that:

- The variable \(x_3 \) does not correspond to a pivot variable.
- The pivots are assigned to other variables (e.g., \(x_1 \) and \(x_2 \)), meaning \(x_3 \) remains free.
- The system is consistent, i.e., no contradictions arise.

Step-by-Step Construction

1. Identify the pivot positions:
- For a 3x4 matrix, typical pivot positions are in columns 1 and 2.
- Column 3 (corresponding to \(x_3 \)) will not contain a pivot, making \(x_3 \) a free

variable.

2. Create the matrix in RREF with pivots in columns 1 and 2:

\[
A = \begin{bmatrix}
1 & 0 & & \\
0 & 1 & & \\
0 & 0 & 0 & \\
\end{bmatrix}
\]

- The asterisks () denote arbitrary numbers, as long as they satisfy RREF conditions.

3. Set the entries to ensure that the third variable \(x_3 \) is free:

\[
A = \begin{bmatrix}
1 & 0 & 2 & | & 3 \\
0 & 1 & -1 & | & 4 \\
0 & 0 & 0 & | & 5 \\
\end{bmatrix}
\]

Note: The last row indicates inconsistency if the right side is non-zero when the entire row is zero on the left. To keep the system consistent, the last row's right side should be zero:

\[
A = \begin{bmatrix}
1 & 0 & 2 & | & 3 \\
0 & 1 & -1 & | & 4 \\
0 & 0 & 0 & | & 0 \\
\end{bmatrix}
\]

This matrix is in RREF, with a free variable \(x_3 \).

Formulating the System and Deriving the General Solution

Corresponding System of Equations

From the matrix:

$$
\begin{bmatrix}
1 & 0 & 2 & | & 3 \\
0 & 1 & -1 & | & 4 \\
0 & 0 & 0 & | & 0 \\
\end{bmatrix}
$$

we get the system:

$$
\begin{cases}
x_1 + 2x_3 = 3 \\
x_2 - x_3 = 4 \\
0 = 0 \quad \text{(no restriction)} \\
\end{cases}
$$

Expressing Leading Variables in Terms of Free Variables

- $x_1 = 3 - 2x_3$
- $x_2 = 4 + x_3$
- x_3 is free and can be any real number.

General Solution:

$$
\boxed{
\begin{cases}
x_1 = 3 - 2t \\
x_2 = 4 + t \\
x_3 = t \\
\end{cases}
\quad \text{where } t \in \mathbb{R}
}
$$

This parametric form clearly shows that x_3 is a free variable, and the solutions form a line in $\mathbb{R}^3$.

Implications and Applications of Matrices with Free Variables

Understanding the Solution Space

- When a system has free variables, it indicates infinitely many solutions.
- The free variables correspond to dimensions of the solution space.
- Visualizing these solutions helps in understanding the geometric interpretation of the system.

Applications in Science and Engineering

- Computer Graphics: Parameterizing lines and planes.
- Engineering: Analyzing systems with constraints and redundancies.
- Data Science: Dimensionality reduction and understanding dependencies among variables.
- Optimization Problems: Identifying degrees of freedom in solutions.

Key Takeaways for Students and Practitioners

- Recognize the importance of pivots and free variables in matrix solutions.
- Master the process of converting matrices to RREF.
- Understand the significance of free variables in the context of solution spaces.
- Be able to construct matrices with specific properties, such as designated free variables.

Summary and Final Remarks

Constructing a 3x4 matrix in reduced echelon form with a free variable x_3 requires strategic placement of pivots and careful design of the matrix entries. This structure ensures the variable x_3 remains free, resulting in an infinite set of solutions described parametrically. Recognizing the role of free variables enhances understanding of the solution space, which is fundamental in applications across science, engineering, and mathematics.

Mastering these concepts not only aids in solving systems of linear equations but also provides insight into the geometric and algebraic nature of linear systems. Whether you are a student learning linear algebra or a professional applying these principles, understanding how to manipulate matrices to reveal solutions is an essential skill.

Additional Resources:

- Linear Algebra Textbooks (e.g., "Linear Algebra and Its Applications" by Gilbert Strang)
- Online Matrix Calculators for Practice

- Tutorials on Row Reduction and Matrix Transformations
- Video Lectures on Solution Spaces and Free Variables

Keywords for SEO Optimization:

- 3x4 matrix in reduced echelon form
- Free variable in linear systems
- Constructing matrices with free variables
- General solution of linear equations
- Reduced row echelon form (RREF)
- Solving systems with free variables
- Matrix solution space analysis
- Linear algebra tutorials

Frequently Asked Questions

What are the key characteristics of a 3x4 matrix in reduced echelon form with a free variable X3?

A 3x4 matrix in reduced echelon form with a free variable X3 has leading 1's in pivot positions, zeros below and above these pivots, and at least one non-pivot column corresponding to the free variable X3, indicating infinitely many solutions.

How do you identify the free variable X3 in the reduced echelon form of a matrix?

In the reduced echelon form, the free variable X3 corresponds to a column without a leading 1 (pivot), meaning it can take any value, and the other variables are expressed in terms of X3.

What is the general solution form of a system represented by a 3x4 matrix with a free variable X3?

The general solution expresses the basic variables as functions of the free variable X3, typically written as parametric equations where X3 is a parameter, e.g., X1 = ... + cX3, X2 = ... + dX3, and X3 is free.

Can you provide an example of a 3x4 matrix in reduced echelon form with X3 as a free variable?

Yes. For example,
[[1, 0, 0, a],
[0, 1, 0, b],
[0, 0, 0, 1]]

This matrix shows X1 and X2 as leading variables and X3 as free, with X4 as a basic variable depending on X3.

How do you interpret the solution when the matrix has a free variable like X3?

The presence of a free variable indicates infinitely many solutions, with the solutions forming a parametric family where X3 can be any real number, and other variables depend on X3.

What steps are involved in finding a matrix in reduced echelon form with a free variable X3 for a given system?

The steps include writing the augmented matrix, applying Gaussian elimination to reach row echelon form, then Gauss-Jordan elimination to achieve reduced echelon form, ensuring X3's column has no pivot, thus identifying it as free.

Why is identifying free variables like X3 important in solving systems of linear equations?

Identifying free variables helps determine whether the system has a unique solution, infinitely many solutions, or no solution, and it allows expressing the general solution parametrically.

Additional Resources

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Introduction: Understanding the Context of the Problem

In the realm of linear algebra, the study of systems of linear equations is fundamental. These systems are often represented in matrix form, which provides a powerful and compact way to analyze their solutions. When working with matrices, a common goal is to find solutions to the corresponding system of equations, especially when the system is underdetermined or has free variables.

The problem at hand involves finding a specific matrix A of size 3×4, expressed in reduced echelon form, with the particular condition that one of its variables, x_3, is a free variable. This scenario implies that the associated system of linear equations has infinitely many solutions, parameterized by the free variable, and the structure of the matrix must reflect this.

Understanding how to construct such a matrix is crucial for students and practitioners alike, as it deepens comprehension of concepts such as row operations, echelon forms, free variables, parametric solutions, and the interpretation of matrices in solving linear systems. This article aims to explore this problem comprehensively, explaining the theoretical background, the step-by-step process for constructing the matrix, and the implications of the solution.

Foundational Concepts in Linear Algebra

Matrix Representation of Linear Systems

Linear systems can be succinctly represented in matrix form as $A\mathbf{x} = \mathbf{b}$, where:

- A is an $m \times n$ matrix containing the coefficients.
- $\mathbf{x}$ is an $n \times 1$ vector of variables.
- $\mathbf{b}$ is an $m \times 1$ vector of constants.

In our case, A is a 3×4 matrix, indicating three equations with four variables, often denoted as x_1, x_2, x_3, x_4.

Reduced Row Echelon Form (RREF)

A matrix is in reduced row echelon form if it satisfies the following conditions:

1. All leading entries (the first non-zero number from the left in each row) are 1 (called leading 1s).
2. Each leading 1 is the only non-zero entry in its column.
3. The leading 1 in each row appears to the right of the leading 1 in the previous row.
4. Rows consisting entirely of zeros are at the bottom of the matrix.

Transforming a matrix into RREF involves elementary row operations: row swapping, scaling rows, and adding multiples of one row to another.

Free Variables and Pivot Positions

In the context of solutions:

- Pivot positions correspond to variables that can be expressed explicitly in terms of other variables.
- Non-pivot columns correspond to free variables, which can take arbitrary values, leading to parametric solutions.

In our problem, the variable x_3 is a free variable, implying that its column in the RREF matrix is a non-pivot column.

Designing the Matrix A: Structural Considerations

Matrix Size and Its Implications

Given that A is a 3×4 matrix, it has 3 equations and 4 variables. This setup naturally suggests that the system is potentially underdetermined, allowing for free variables, provided the rank of A is less than 4.

Since the goal is to have x_3 as a free variable, the third variable's column in the RREF matrix must be a non-pivot column. The other variables may or may not be pivot variables, but to maintain generality and clarity, one common approach is to assign pivot positions to x_1 and x_2, and leave x_3 free, with x_4 possibly as a pivot or free as needed.

Choosing Pivot and Free Variables

To ensure x_3 is free:

- The column corresponding to x_3 must not contain a leading 1 in the RREF.
- The columns for pivot variables should have leading 1s, with zeros elsewhere in their columns.

A typical assignment:

- Pivot columns: x_1 and x_2
- Free column: x_3
- x_4 could be a free variable or a pivot, depending on the desired solution structure.

For simplicity, let's assume:

- Pivot variables: x_1, x_2
- Free variables: x_3, x_4

However, since the problem specifies only x_3 as a free variable, we will design the matrix with only x_3 free, and x_4 as a pivot variable to maintain the total of three pivot positions (since the matrix has only 3 rows).

Constructing the Reduced Echelon Form Matrix A

Step 1: Establishing the Pivot Positions

Given the assumptions:

- x_1 and x_2 are pivot variables.
- x_3 is free.
- x_4 is a pivot variable (to ensure the rank is 3, matching the number of rows).

The pivot columns could be:

- Column 1: x_1
- Column 2: x_2
- Column 4: x_4

And column 3 (x_3) remains free.

Step 2: Formulating the RREF Matrix

In RREF, the matrix A should look like:

$$
A = \begin{bmatrix}
1 & 0 & & 0 \\
0 & 1 & & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

However, to ensure x_3 remains free, the third column must not contain a leading 1, i.e., it should be a zero column.

Therefore, the matrix in RREF form could be:

\[
A = \begin{bmatrix}
1 & 0 & 0 & a \\

0 & 1 & 0 & b \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

But this configuration would imply that \(x_4\) is a pivot variable, which is acceptable. Alternatively, for simplicity, one can construct:

\[
A = \begin{bmatrix}
1 & 0 & 0 & c \\
0 & 1 & 0 & d \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

which indicates that:

- \(x_1 =\) expressed in terms of parameters,
- \(x_2 =\) expressed similarly,
- \(x_3\) is free (since its column has no leading 1),
- \(x_4\) can be a free variable or a dependent variable depending on the constants.

Step 3: Finalizing the Matrix with Constants

To fully specify the matrix, assign particular values to the free variables and constants. For the purpose of the example, let's choose:

\[
A = \begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

This matrix is in RREF, with pivot positions at columns 1 and 2. The third variable \(x_3\) does not have a leading 1 in its column, making it a free variable.

Interpreting the Solutions and the Role of Free Variable \(x_3\)

Parametric Form of the Solution

The corresponding system of equations represented by the matrix:

$$
\begin{cases}
x_1 + 2x_4 = \text{constant}_1 \\
x_2 - x_4 = \text{constant}_2 \\
0 = 0
\end{cases}
$$

Assuming homogeneous equations (right-hand side equals zero), the solutions are:

$$
x_1 = -2x_4 + s_1
$$
$$
x_2 = x_4 + s_2
$$
$$
x_3 = t
$$
$$
x_4 = \text{free parameter}
$$

with t, x_4 as arbitrary parameters, illustrating how x_3 remains free.

Implications of the Free Variable

Having x_3 as a free variable means the solution set forms an affine subspace of $\mathbb{R}^4$, with the parametric form depending on the free variables t (for x_3) and x_4.

This flexibility is essential in understanding the solution space's dimensionality, which, in this case, is at least 2 (since two

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