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Introduction to Linear Combinations

Understanding the Concept

A fundamental concept in linear algebra is the idea of expressing a vector as a linear combination of other vectors. Specifically, given vectors U_1, U_2, U_3, and a vector V, the question is whether V can be written in the form:

$$
V = a_1 U_1 + a_2 U_2 + a_3 U_3
$$

where a_1, a_2, a_3 are scalars (real numbers in most cases). If such scalars exist, then V is said to lie in the span of U_1, U_2, U_3. Otherwise, it is impossible to express V as their linear combination.

Determining If V Can Be Expressed as a Linear Combination

Step 1: Set Up the Equation

To check if V can be written as a linear combination of U_1, U_2, U_3, set up the vector equation:

$$
V = a_1 U_1 + a_2 U_2 + a_3 U_3
$$

This can be written component-wise as a system of linear equations. For example, if the vectors are in $\mathbb{R}^n$:

$$
\begin{cases}
V_1 = a_1 U_{1,1} + a_2 U_{2,1} + a_3 U_{3,1} \\
V_2 = a_1 U_{1,2} + a_2 U_{2,2} + a_3 U_{3,2} \\
\vdots \\
V_n = a_1 U_{1,n} + a_2 U_{2,n} + a_3 U_{3,n}
\end{cases}
$$

This forms a system of n equations with 3 unknowns a_1, a_2, a_3.

Step 2: Formulate the Matrix Equation

Express the system in matrix form:

$$
\begin{bmatrix}
U_{1,1} & U_{2,1} & U_{3,1} \\
U_{1,2} & U_{2,2} & U_{3,2} \\
\vdots & \vdots & \vdots \\
U_{1,n} & U_{2,n} & U_{3,n}
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix}
=
\begin{bmatrix}
V_1 \\
V_2 \\
\vdots \\
V_n
\end{bmatrix}
$$

Let’s denote:
- A as the $n \times 3$ matrix comprising the vectors U_1, U_2, U_3 as columns.
- $\mathbf{a}$ as the vector of unknown scalars.
- $\mathbf{V}$ as the target vector.

The question reduces to whether the system $A \mathbf{a} = \mathbf{V}$ has a solution.

Analyzing the System for Solvability

Step 1: Check the Rank of the Matrix A

The key to solving the system lies in the rank of matrix A:

- If $\text{rank}(A) = \text{rank}([A | \mathbf{V}])$, then the system is consistent, and V can be expressed as a linear combination of U_1, U_2, U_3.
- If $\text{rank}(A) \neq \text{rank}([A | \mathbf{V}])$, then the system is inconsistent, and expressing V as a linear combination is impossible.

Note: The rank of a matrix is the maximum number of linearly independent columns or rows.

Step 2: Use Row Reduction to Determine Consistency

Perform Gaussian elimination or row reduction on the augmented matrix:

$$
[A | \mathbf{V}]
$$

- Transform the matrix into row echelon form.
- Check for any row indicating inconsistency, such as:

$$
0 \quad 0 \quad 0 \quad | \quad c \neq 0
$$

which indicates no solution.

Example: Applying the Method

Given Vectors

Suppose in $\mathbb{R}^3$:

$$
U_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad
U_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \quad
U_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}
$$

and

\[
V = \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix}
\]

Set Up the System

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
2 & 3 & 4
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix}
=
\begin{bmatrix}
2 \\
3 \\
9
\end{bmatrix}
\]

Row Reduction

Perform Gaussian elimination:

1. The first row is already leading with 1 in the first position.
2. Use row 1 to eliminate the first element in row 3:

\[
R_3 \leftarrow R_3 - 2 R_1
\]

\[
\begin{bmatrix}
1 & 0 & 1 & | & 2 \\
0 & 1 & 1 & | & 3 \\
0 & 3 & 2 & | & 5
\end{bmatrix}
\]

3. Next, eliminate the second element in row 3:

\[

$$
R_3 \leftarrow R_3 - 3 R_2
$$

$$
\begin{bmatrix}
1 & 0 & 1 & | & 2 \\
0 & 1 & 1 & | & 3 \\
0 & 0 & -1 & | & -4
\end{bmatrix}
$$

4. Back-substitute to find the scalars:

- From row 3:

$$
-1 a_3 = -4 \Rightarrow a_3 = 4
$$

- From row 2:

$$
a_2 + a_3 = 3 \Rightarrow a_2 + 4 = 3 \Rightarrow a_2 = -1
$$

- From row 1:

$$
a_1 + a_3 = 2 \Rightarrow a_1 + 4 = 2 \Rightarrow a_1 = -2
$$

Thus, the solution exists, and V can be written as:

$$
V = -2 U_1 - 1 U_2 + 4 U_3
$$

When Is It Impossible?

Situations Leading to Impossibility

- The system is inconsistent, meaning no combination of scalars a_1, a_2, a_3 satisfies the equations.
- The vectors U_1, U_2, U_3 are linearly dependent, but V lies outside their span.
- The dimension of the span of U_1, U_2, U_3 is less than 3, and V is outside this span.

Implication of Dependence

If the vectors are linearly dependent, then their span is a subspace of lower dimension. To determine whether V is in this subspace, check whether V can be written as a linear combination of the independent vectors.

Summary and Practical Steps

- Set up the linear system based on the components of V and U_1, U_2, U_3.
- Form the matrix A with U_1, U_2, U_3 as columns.
- Augment A with V and perform row reduction.
- Compare ranks to determine if the system is consistent.
- If consistent, find the scalars a_1, a_2, a_3; if not, conclude that it is impossible.
