A Hemispherical Shaped Tank Has A Radius Of 10 Ft. If The Tank Is Completely Full, Find The Work Done

A Hemispherical Shaped Tank Has A Radius Of 10 Ft. If The Tank Is Completely Full, Find The Work Done is a classic problem in fluid mechanics and physics that combines geometric reasoning with principles of work and energy. When dealing with tanks filled with liquid, such as water, understanding the work required to pump out or move the fluid can provide critical insights for engineering applications, including design and efficiency calculations. In this article, we will explore the process step-by-step, covering the geometric properties of the hemispherical tank, the principles involved in calculating work, and a detailed solution to find the work done to empty the tank completely.

Understanding the Geometry of the Hemispherical Tank

The Shape and Dimensions

A hemispherical tank is essentially half of a sphere. Given that the radius of the tank is 10 feet, the geometric properties are well-defined:

- Radius (r): 10 ft
- Diameter (d): 20 ft
- Volume of the tank (V): The volume of a hemisphere

Understanding these dimensions is fundamental because the volume and the surface area calculations directly impact the work done calculations.

Volume of the Hemisphere

The volume of a sphere is given by:

$$ V_{sphere} = \frac{4}{3} \pi r^3 $$

Since our tank is a hemisphere:

$$ V_{hemisphere} = \frac{1}{2} \times \frac{4}{3} \pi r^3 = \frac{2}{3} \pi r^3 $$

Plugging in r = 10 ft:

$$ V_{hemisphere} = \frac{2}{3} \pi (10)^3 = \frac{2}{3} \pi \times 1000 = \frac{2000}{3} \pi \, $$

\text{cubic feet} \]

This total volume is essential because it represents the complete amount of fluid to be pumped out, and the work calculation involves moving this fluid to a certain height.

Principles of Work in Fluid Pumping

The Concept of Work Done to Pump Fluids

Work, in physics, is defined as force applied over a distance. When pumping fluid from a tank, the work done is equal to the energy required to lift the fluid from its initial position to the outlet or to a specified height, typically ground level.

The key factors include:

- The weight of the fluid being moved
- The vertical distance the fluid must be lifted
- The variable volume of fluid at different heights within the tank

In a filled tank, fluid at higher levels requires less work to move than fluid at lower levels, because it is closer to the outlet.

Using Integration to Calculate Work

Since the volume of fluid at different heights varies, and the height each “slice” of fluid must be lifted depends on its position within the tank, calculus becomes a powerful tool to compute the total work.

The general approach involves:

1. Dividing the tank into thin horizontal slices
2. Calculating the volume of each slice
3. Determining the weight of each slice
4. Calculating the work to lift each slice to the outlet (assumed at the top or bottom)
5. Integrating over the entire height to find total work

Step-by-Step Solution to Find the Work Done

Setting Up the Coordinate System

To facilitate the integration, we establish a coordinate system:

- The center of the hemisphere at the origin (0,0,0)
- The vertical axis (z-axis) going upwards
- The top of the hemisphere at z = r = 10 ft
- The bottom at z = -r = -10 ft

Since the tank is a hemisphere, the surface satisfies the equation:

$$ x^2 + y^2 + z^2 = r^2 $$

For calculations, we consider horizontal slices at a height z, each with a small thickness dz.

Expressing the Cross-Sectional Area

At a given height z, the cross-sectional circle has radius:

$$ \rho(z) = \sqrt{r^2 - z^2} $$

The area of this cross-section:

$$ A(z) = \pi \rho(z)^2 = \pi (r^2 - z^2) $$

The volume of a thin slice at height z with thickness dz:

$$ dV = A(z) \, dz = \pi (r^2 - z^2) \, dz $$

Calculating the Mass and Weight of Each Slice

Assuming the fluid density is ρ_{fluid} (say, for water, approximately 62.4 lb/ft^3):

$$ \text{Mass} (dm) = \rho_{fluid} \times dV $$

$$ \text{Weight} (dW_{weight}) = dm \times g $$

Since the density is in lb/ft^3 and g is incorporated into the density (for standard gravity), we can directly use the weight per unit volume:

$$ \text{Weight of slice} = \rho_{fluid} \times dV $$

Determining the Work to Lift Each Slice

The work to lift a small volume of fluid from height z to the outlet depends on the height:

- If the outlet is at the top (z = r), the lift height for the slice at height z is:

$$ h(z) = r - z $$

- The work for each slice:

$$ dW = \text{Weight of slice} \times h(z) $$

$$ dW = \rho_{fluid} \times \pi (r^2 - z^2) \times (r - z) \, dz $$

Integrating to Find Total Work

The total work W is obtained by integrating from the bottom of the tank at z = -r to the top at z = r:

$$ W = \int_{z=-r}^{z=r} \rho_{fluid} \pi (r^2 - z^2) (r - z) \, dz $$

Substituting r = 10 ft:

$$ W = \rho_{fluid} \pi \int_{-10}^{10} (100 - z^2)(10 - z) \, dz $$

Computing the Integral

Simplifying the Integrand

Expand the integrand:

$$ (100 - z^2)(10 - z) = 100 \times (10 - z) - z^2 \times (10 - z) $$

$$ = 1000 - 100z - 10z^2 + z^3 $$

Thus, the integral becomes:

$$ W = \rho_{fluid} \pi \int_{-10}^{10} (1000 - 100z - 10z^2 + z^3) \, dz $$

Calculating the Integral Term-by-Term

Integrate each term:

1. $\int 1000 \, dz = 1000z$
2. $\int -100z \, dz = -50z^2$
3. $\int -10z^2 \, dz = -\frac{10}{3} z^3$
4. $\int z^3 \, dz = \frac{1}{4} z^4$

Evaluate from z = -10 to z = 10:

\[

W = \rho_{fluid} \pi \left[\left(1000z - 50z^2 - \frac{10}{3}z^3 + \frac{1}{4}z^4 \right) \bigg|_{-10}^{10} \right]
\]

Calculate at z = 10:

\[
1000 \times 10 - 50 \times 100 - \frac{10}{3} \times 1000 + \frac{1}{4} \times 10000
= 10,000 - 5,000 - \frac{10,000}{3} + 2,500
\]

Simplify:

\[
(10,000 + 2,500) - 5,000 - \frac{10,000}{3} = 12,500 - 5,000 - 3,333.\overline{3} = 7,500 - 3,333.\overline{3} = 4,166.\overline{6}
\]

Now at z = -10:

\[
1000 \times (-10) - 50 \times 100 - \frac{10}{3} \times (-1000) + \frac{1}{4} \times 10,000
\]

\[
-10,000 - 5,000 + \frac{10,000}{3} + 2,500
\]

Simplify:

\[
(-10,000 - 5,000 + 2,500) + \frac{10,000}{3} = (-12,500) + 3,333.\overline{3} = -9,166.\overline{6}
\]

Subtract the two:

\[
W = \rho_{fluid} \pi \left(4,166.\overline{

Frequently Asked Questions

What is the volume of the hemispherical tank with a radius of 10 ft when it is completely full?

The volume of the hemispherical tank is $(2/3)\pi r^3$. Substituting r = 10 ft, Volume = $(2/3)\pi(10)^3$ = $(2/3)\pi(1000) \approx 2094.4$ cubic feet.

How do you set up the integral to calculate the work done in pumping out the liquid from the full hemispherical tank?

The work done is found by integrating the weight of each horizontal layer of liquid times the distance it must be pumped. This involves integrating over the height from the bottom to the top of the tank, considering the cross-sectional area at each level.

What is the density of the liquid assumed in calculating the work done, and how does it affect the result?

Typically, the density is assumed based on the liquid in the tank (e.g., water = 62.4 lb/ft^3). The work done is directly proportional to the density; higher density liquids require more work to pump out.

How do you determine the limits of integration when calculating work for a hemispherical tank?

The limits of integration are from the bottom (0 ft) to the top (10 ft) of the hemisphere, representing the vertical position of each thin layer of liquid being pumped out.

What is the formula for the work done in pumping the liquid out of the hemispherical tank?

Work = ∫ (weight of a thin slice) × (distance to pump), which simplifies to W = ∫_{0}^{h} (density × area at height y) × (distance from y to the outlet), integrated over the height of the liquid.

Can you provide a simplified expression for calculating the work done in this scenario?

Yes, after setting up the integral considering the cross-sectional area at height y, the work done can be expressed as an integral involving the radius, density, and geometric relationships, often resulting in a formula involving π, r, and y.

Why is it important to consider the shape of the tank when calculating the work done?

Because the shape determines the cross-sectional area at each height, which affects the volume and weight of the liquid at each layer, thus influencing the total work required to pump out the entire contents.

Additional Resources

A Hemispherical Shaped Tank Has A Radius Of 10 Ft. If The Tank Is Completely Full, Find The Work Done

Understanding the problem of calculating the work done to fill a hemispherical tank involves an intriguing combination of geometry, calculus, and physics principles. At its core, this problem presents a real-world application of volume and surface area calculations, fluid mechanics, and the concept of work in physics — specifically, the work required to lift a fluid against gravity to fill a tank. This comprehensive review aims to dissect each aspect of the problem, explore the mathematical framework necessary for a solution, and discuss the practical implications and features of such calculations.

Understanding the Problem Context

The problem involves a hemispherical tank with a radius of 10 feet, which is completely filled with a liquid (most likely water). The goal is to determine the total work done in filling the tank. This is a classic problem in fluid mechanics and calculus-based physics, often encountered in engineering disciplines.

Key Elements of the Problem:
- Shape of the tank: Hemispherical
- Radius of the tank: 10 feet
- Filling condition: Completely full
- Quantity to find: Work done in filling the tank

The problem implicitly assumes certain physical parameters:
- The density of the fluid (commonly water, which has a density of approximately 62.4 lb/ft^3)
- The gravitational acceleration (standard gravity, $g \approx 32.2$ ft/sec^2)

Geometry of the Hemispherical Tank

Basic Geometrical Properties

A hemisphere is half of a sphere, and its volume and surface area are well-known geometrical quantities:

- Radius (r): 10 ft
- Diameter: 20 ft
- Volume of Hemisphere: $V = \frac{2}{3} \pi r^3$

Calculating the total volume:

$$
V = \frac{2}{3} \pi (10)^3 = \frac{2}{3} \pi \times 1000 = \frac{2000}{3} \pi \text{ ft}^3 \approx 2094.4 \text{ ft}^3
$$

This volume indicates the total amount of fluid the tank can hold when full.

Coordinate System and Cross-Sections

To analyze the work done in filling the tank, especially with calculus, a coordinate system is typically adopted:

- Place the origin at the center of the flat face of the hemisphere at the bottom.
- The hemisphere extends upward in the positive y-direction.
- The equation of the hemispherical surface becomes:

$$
x^2 + z^2 + y^2 = r^2, \quad y \geq 0
$$

This setup simplifies the problem by allowing us to consider thin horizontal slices at different heights y, which are easier to work with when integrating.

Physics of Work Done in Filling the Tank

Definition of Work in Fluid Filling

Work is defined as force times distance. When filling a tank, the work done involves lifting each infinitesimal volume of fluid from the source to its final position within the tank. For each small volume element, the work depends on:

- The weight of the fluid: $\text{Weight} = \text{Volume} \times \text{Density}$
- The vertical distance it must be lifted: from the source (assumed at the bottom or a reference point) to the height where the volume is deposited

Assumption: The fluid is poured from above at a constant height or from a point at the top, which simplifies the calculation of the lifting distance. Alternatively, if the fluid is being pumped from a source at ground level, the lifting distance varies depending on the position within the tank.

Work Calculation Approach

The typical approach involves integrating the work over all thin slices of the fluid:

$$
W = \int \text{(force)} \times \text{(distance)}
$$

For a fluid element at height y, the steps are:

1. Find the volume of the thin layer at height y.
2. Calculate its weight.
3. Determine the vertical distance it must be lifted.
4. Integrate over the entire height of the hemisphere.

Mathematical Setup for the Calculation

Volume of a Horizontal Slice

At a height y, the cross-section of the hemispherical tank is a circle with radius:

$$
x = \sqrt{r^2 - y^2}
$$

The differential volume element dV:

$$
dV = \text{Area of cross-section} \times dy = \pi x^2 dy = \pi (r^2 - y^2) dy
$$

The total volume is obtained by integrating dV over y from 0 to r.

Calculating the Work

Assuming the fluid is lifted from the bottom (at $y=0$) to the top (at $y=r$), the work to lift each slice is:

$$
dW = \text{weight} \times \text{distance}
$$

Since the weight of each slice:

$$
dW = \text{density} \times dV \times g \times (\text{height it is lifted})
$$

If the fluid is being lifted from the bottom (or a reference point), the lifting distance for the slice at height y is:

$$
\text{distance} = y
$$

Putting it together:

$$
dW = \rho \, g \, (\pi (r^2 - y^2) dy) \times y
$$

The total work:

$$
W = \int_{0}^{r} \rho g \pi (r^2 - y^2) y \, dy
$$

Step-by-Step Calculation

Parameter Values

- Radius $r = 10$ ft
- Density of water $\rho = 62.4$ lb/ft^3
- Gravitational acceleration $g = 32.2$ ft/sec^2

Note: In imperial units, the weight of the fluid per unit volume is already incorporated in $\rho \times g$.

Integral Setup

$$
W = \pi \rho g \int_{0}^{r} (r^2 - y^2) y \, dy
$$

Expand the integrand:

$$
(r^2 - y^2) y = r^2 y - y^3
$$

Therefore:

$$
W = \pi \rho g \left(r^2 \int_{0}^{r} y \, dy - \int_{0}^{r} y^3 \, dy \right)
$$

Calculating the integrals:

$$
\int_{0}^{r} y \, dy = \frac{r^2}{2}
$$

$$
\int_{0}^{r} y^3 \, dy = \frac{r^4}{4}
$$

Putting these back into the expression:

$$
W = \pi \rho g \left(r^2 \times \frac{r^2}{2} - \frac{r^4}{4} \right) = \pi \rho g \left(\frac{r^4}{2} - \frac{r^4}{4} \right)
$$

Simplify:

$$
W = \pi \rho g \times \frac{r^4}{4}
$$

Plugging in the numerical values:

$$
W = \pi \times 62.4 \, \text{lb/ft}^3 \times 32.2 \, \text{ft/sec}^2 \times \frac{(10)^4}{4}
$$

Calculate r^4:

$$
(10)^4 = 10,000
$$

Thus,

$$
W = \pi \times 62.4 \times 32.2 \times \frac{10,000}{4}
$$

Simplify:

$$
W = \pi \times 62.4 \times 32.2 \times 2,500
$$

Calculate step-by-step:

- $62.4 \times 32.2 \approx 2008.$
- $2008 \times 2,500 = 2008 \times 2,500 \approx 5,020,000$

Finally,

$$
W \approx \pi \times 5,020,000 \approx 3.1416 \times 5,020,000 \approx 15,750,000 \text{ ft-lb}
$$

Therefore, the work done in filling the hemispherical tank is approximately 15.75 million foot-pounds.

Features, Pros, and Cons of the Calculation Method

Features:
- Utilizes calculus to handle the variable volume and lifting distances.
- Incorporates physical parameters like fluid density and gravity.
- Extensible to different geometries or fluid types.

Pros:
- Precise and mathematically rigorous.
- Can accommodate complex shapes with appropriate adjustments.
- Provides insight into the energy requirements for fluid handling.

Cons:

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