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Introduction

In probability theory, experiments involving repeated random trials are fundamental to understanding how likelihoods behave under various conditions. One intriguing scenario is when an experiment involves rolling a fair die multiple times until a specific outcome occurs a fixed number of times. This type of problem is not only academically interesting but also has practical applications in fields like quality control, game theory, and statistical modeling.

In this article, we explore a specific experiment: tossing a fair six-sided die repeatedly until the number 5 has appeared exactly six times. The core question we aim to answer is: What is the probability that a particular event occurs during this process? For example, we might want to find the probability that the total number of rolls needed is a certain value, or that the number 5 appears a specific number of times within a certain number of trials.

Understanding this problem involves delving into concepts such as the negative binomial distribution, geometric distribution, and the properties of independent Bernoulli trials. We will analyze the problem systematically, derive formulas, and explore various probability scenarios associated with this experiment.

Setting Up the Problem

The Experiment

The experiment involves:

- Tossing a fair die repeatedly.
- Each roll is independent.
- The probability of rolling a 5 on any single roll is $p = \frac{1}{6}$.
- The probability of not rolling a 5 is $q = 1 - p = \frac{5}{6}$.

The process continues until we observe the number 5 exactly 6 times, which we can interpret as a stopping rule based on the count of successes (rolling a 5).

Key Concepts

Before proceeding, it’s essential to understand the core probability concepts involved:

- Bernoulli Trials: Each die roll can be considered a Bernoulli trial with two outcomes—success (rolling a 5) and failure (not rolling a 5).
- Number of Successes: The total number of times 5 appears during the process.
- Stopping Rule: The process ends once the total successes reach 6.

Formalizing the Question

The main questions we aim to answer include:

1. What is the probability that exactly n rolls are needed to observe 6 occurrences of 5?

2. What is the probability that the total number of rolls until the 6th 5 appears is less than or equal to n?

3. What is the distribution of the number of rolls needed to reach 6 successes?

4. What is the probability that the 6th 5 occurs at a specific roll number n?

These questions can be approached by understanding the negative binomial distribution, which models the number of trials needed to achieve a fixed number of successes in independent Bernoulli trials.

The Negative Binomial Distribution

Definition

The negative binomial distribution describes the probability of having r successes after n trials, with the condition that the r-th success occurs on trial n.

Mathematically, the probability that the r-th success occurs exactly on trial n (meaning the process ends at the n-th trial) is given by:

$$
P(N = n) = \binom{n-1}{r-1} p^{r} q^{n - r}
$$

where:

- N is the random variable representing the trial number of the r-th success.
- r is the total number of successes desired (here, 6).
- p is the probability of success on each trial ($\frac{1}{6}$ for rolling a 5).
- $q = 1 - p$.

This formula gives the probability that the r-th success occurs exactly on the n-th trial, with the first $n-1$ trials containing exactly $r-1$ successes.

Application to Our Problem

In our case:

- $r = 6$
- $p = \frac{1}{6}$
- $q = \frac{5}{6}$

Thus, the probability that the 6th 5 occurs exactly on the n-th roll is:

$$
\boxed{
P(N = n) = \binom{n-1}{5} \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^{n - 6}
}
$$

for $n \geq 6$, since at least 6 rolls are needed to achieve 6 successes.

Key Probability Calculations

Probability That the 6th 5 Occurs at a Specific Roll n

Using the negative binomial formula, the probability that the process stops exactly at the n-th roll is:

$$
P(N = n) = \binom{n-1}{5} \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^{n - 6}
$$

where:

- $n \geq 6$,
- $\binom{n-1}{5}$ accounts for the ways to arrange 5 successes in the first $n-1$ trials.

Total Probability for the Process

The sum over all possible $n \geq 6$ of these probabilities must be 1, representing certainty that the process will end at some finite trial:

$$
\sum_{n=6}^{\infty} P(N = n) = 1
$$

which confirms that the negative binomial distribution is a valid probability distribution.

Expected Number of Rolls to Achieve 6 Successes

The expected value (mean) of N, the number of rolls needed to get 6 successes, is given by:

$$
E[N] = \frac{r}{p} = \frac{6}{\frac{1}{6}} = 36

\]

This means, on average, it takes 36 rolls to observe 6 occurrences of 5 when rolling a fair die.

Probability Scenarios and Examples

Example 1: Probability That the 6th 5 Occurs Exactly at the 40th Roll

Using the formula:

\[
P(N = 40) = \binom{39}{5} \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^{34}
\]

Calculations involve:

- Computing \(\binom{39}{5} \),
- Raising \(\frac{1}{6} \) to the 6th power,
- Raising \(\frac{5}{6} \) to the 34th power.

This gives a specific probability value indicating how likely it is for the process to conclude exactly at the 40th roll.

Example 2: Probability That the 6th 5 Occurs Within 50 Rolls

This involves calculating the cumulative distribution function (CDF):

\[
P(N \leq 50) = \sum_{n=6}^{50} P(N = n)
\]

which sums the probabilities from the minimum number of rolls (6) up to 50.

Example 3: Probability That the 6th 5 Occurs Before the 36th Roll

Similarly:

\[
P(N < 36) = \sum_{n=6}^{35} P(N = n)
\]

providing insights into the likelihood of achieving the 6 successes relatively quickly.

Advanced Considerations

Variance of the Number of Rolls

The variance of N, the total number of rolls needed to get 6 successes, is given by:

$$
\text{Var}(N) = \frac{r q}{p^2} = \frac{6 \times \frac{5}{6}}{\left(\frac{1}{6}\right)^2} = 6 \times \frac{5}{6} \times 36 = 180
$$

This high variance indicates significant variability around the mean of 36 rolls.

Alternative Perspectives: Binomial Distribution

While the negative binomial describes the distribution of the trial count until success, the binomial distribution can describe the number of successes in a fixed number of trials. For example, if we fix the number of rolls n, the probability of getting exactly k 5s is:

$$
P(\text{exactly } k \text{ successes in } n \text{ trials}) = \binom{n}{k} p^k q^{n - k}
$$

However, our focus is on the stopping time distribution, which aligns with the negative binomial.

Practical Applications and Real-World Implications

Understanding the probability distribution of the number of trials until a fixed number of successes is achieved has numerous applications:

- Quality Control: Estimating how many inspections are needed before a certain number of defects are detected.
- Game Design: Analyzing the expected number of turns until a specific event occurs.
- Risk Assessment: Calculating the likelihood of achieving a set number of failures or successes within a given timeframe.
- Biology: Modeling the number of experiments until a certain number of positive responses are observed.

Conclusion

The problem of determining the probability that an experiment involving

Frequently Asked Questions

What is the probability that the die will land on 5 exactly 6 times before any other number appears 6 times in the

experiment?

The probability that the 5 occurs exactly 6 times before any other number reaches 6 times is given by a negative binomial distribution. Since the die is fair, the probability of rolling a 5 is 1/6. The probability that the first 6 occurrences of 5 happen before any other number reaches 6 is complex and involves analyzing competing geometric processes, but generally, it can be approached by considering the negative binomial distribution for the 5s and the stopping condition when another number reaches 6.

How do you model the process of tossing a fair die until 5 occurs 6 times in terms of probability distribution?

This process can be modeled using a negative binomial distribution, where each trial is a die roll, and success is rolling a 5. The negative binomial gives the probability of observing the 6th success (the 6th time 5 appears) on a certain trial. However, since the process stops when any other number appears 6 times, the model becomes a more complex multi-state process involving competing negative binomial distributions for each face.

What is the significance of the 'stopping condition' in calculating the probability in this experiment?

The stopping condition — that the experiment ends when 5 occurs 6 times or any other number occurs 6 times — introduces a competition between different outcomes. This makes the probability calculation a problem of analyzing absorbing states in a Markov process, where each face has a count, and the process stops when any count reaches 6. It requires calculating the probability that 5 reaches 6 first, considering all possible paths.

Can this problem be simplified by symmetry considerations, and if so, how?

Yes, symmetry can help because all faces of a fair die are equally likely and symmetric. Assuming the process is symmetric with respect to all numbers other than 5, the probability that 5 reaches 6 first can be interpreted as the probability that a particular face (5) reaches its threshold before any other face does. This reduces to a problem of competing 'race' processes with equal probabilities, often leading to simplified calculations based on combinatorial or probabilistic symmetry arguments.

What is the general approach to compute the probability that 5 occurs 6 times first before any other face reaches 6 in this experiment?

The general approach involves modeling the process as a multi-absorbing Markov chain, where each state tracks the counts of each face. The goal is to find the probability that the state where 5 reaches 6 first is reached before any other face reaches 6. This often requires solving a system of linear equations derived from the transition probabilities, leveraging symmetry and combinatorial arguments to simplify the calculations.

Additional Resources

Probability: Unveiling the Intricacies of a Die Tossing Experiment

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Introduction

Imagine a scenario where you are rolling a fair six-sided die repeatedly, and your goal is to observe a specific outcome—namely, the face showing a 5—exactly six times. This seemingly simple experiment opens a window into the fascinating world of probability theory, combinatorics, and stochastic processes. As we delve into this problem, our primary focus will be to determine the probability that a particular event occurs during this experiment, specifically, the probability that the experiment concludes with the sixth occurrence of the face 5.

In this review, we will explore the theoretical underpinnings, dissect the problem into manageable components, and provide a comprehensive analysis of how to approach and solve such probability questions. Whether you are a student, a researcher, or an enthusiast eager to understand the mechanics behind random processes, this article aims to equip you with a detailed understanding of the problem and the methodology to find its solution.

Setting the Stage: Understanding the Experiment

The Experiment's Framework

At its core, the experiment involves:

- Repeated independent trials: Each trial is a single roll of a fair die.
- Equal probability for each outcome: Since the die is fair, each face (1 through 6) has a probability of \(\frac{1}{6}\).
- Stop condition: The process continues until the face 5 has appeared exactly six times.

The question of interest is: What is the probability that a certain event occurs during this process? For example, one natural question is:

- "What is the probability that the process ends with the sixth occurrence of 5?"

or alternatively,

- "What is the probability that, during the process, the sixth 5 appears at a specific point?"

The problem can be formalized further by specifying what event we are analyzing. For clarity, let's consider the most straightforward and common interpretation: the probability that the process terminates exactly when the sixth 5 appears, given the process starts with no prior outcomes.

Conceptual Foundations: The Negative Binomial Distribution

Before diving into calculations, it's crucial to understand the underlying stochastic processes. This problem is tightly linked to the Negative Binomial distribution, which models the number of trials needed to achieve a fixed number of successes in a sequence of independent Bernoulli trials.

Bernoulli Trials and Success Probability

Each die roll can be viewed as a Bernoulli trial with:

- Success: rolling a 5 (probability $p = \frac{1}{6}$)
- Failure: rolling any other number (probability $q = 1 - p = \frac{5}{6}$)

The key question is: How many trials are needed until the sixth success?

Formalizing the Problem

Given the above, the problem reduces to:

> "What is the probability that the sixth success (roll of 5) occurs exactly on the n^{th} trial?"

This is a classic Negative Binomial problem, where the number of trials needed to reach the r^{th} success (here $r=6$) is a random variable.

The Negative Binomial Distribution

The probability that the r^{th} success occurs exactly on the n^{th} trial is:

$$
P(N = n) = \binom{n-1}{r-1} p^{r} q^{n - r}
$$

where:
- N is the trial count at which the r^{th} success occurs;
- p is the success probability ($\frac{1}{6}$);
- $q = 1 - p$;
- r is the total number of successes required (6 in this case);
- $n \geq r$.

This formula accounts for:
- The $\binom{n-1}{r-1}$ ways to arrange the first $r-1$ successes within the first $n-1$ trials;
- The r^{th} success occurs precisely at the n^{th} trial.

Key Probability Computation: The Distribution of the Stopping Time

Given our initial setup, the total probability that the process ends exactly when the sixth 5 appears—meaning the sixth success occurs at some trial $n \geq 6$—can be expressed as:

$$

$$
P(\text{Stopping at } n) = \binom{n - 1}{5} \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^{n - 6}
$$

for all $n \geq 6$.

Note: The total probability that the process ends at some point is 1, since eventually, the process must reach six successes (assuming the process continues until then).

Calculating the Total Probability: The Distribution of the Total Number of Trials

The complete probability distribution over all possible stopping times is:

$$
P(N \geq 6) = 1
$$

and

$$
P(\text{the process ends exactly at } n) = \binom{n - 1}{5} \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^{n - 6}
$$

for each $n \geq 6$.

The Expected Number of Rolls

The expectation of the total number of trials until the sixth success can be derived from properties of the Negative Binomial distribution:

$$
E[N] = \frac{r}{p} = \frac{6}{\frac{1}{6}} = 36
$$

This means, on average, 36 rolls are needed to observe six 5's.

Practical Implications and Probabilities

Knowing the distribution allows us to compute various probabilities:

- Probability that the sixth 5 occurs on exactly the 36th roll:

$$
P(N = 36) = \binom{35}{5} \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^{30}

\]

- Probability that the sixth 5 occurs before or at the 40th roll:

$$
P(N \leq 40) = \sum_{n=6}^{40} \binom{n - 1}{5} \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^{n - 6}
$$

Calculating these probabilities involves summing binomial terms, which can be made more efficient with computational tools or approximations.

Extending the Analysis: Conditional Probabilities and Variations

Beyond the basic probability of stopping at a particular trial, several variations and related questions are of interest:

- What is the probability that the sixth 5 occurs within a certain number of rolls?

This involves cumulative sums of the negative binomial probabilities.

- What is the distribution of the total number of rolls?

As established, it follows a Negative Binomial distribution with parameters $r=6$ and $p=\frac{1}{6}$.

- What if the die were biased?

The entire analysis adapts by replacing $p = \text{probability of rolling a 5}$ with the biased probability.

Practical Applications and Real-World Relevance

While this problem might appear purely theoretical, its implications extend to various fields:

- Quality Control: Modeling the number of inspections until a certain number of defects are found.
- Biology: Understanding the number of trials until a certain number of successes (e.g., cell mutations).
- Gaming and Gambling: Calculating the odds of achieving specific outcomes after a set number of trials.
- Computer Science: Analyzing algorithms that depend on random success events.

Summary and Final Remarks

This detailed analysis of tossing a fair die until 5 occurs six times demonstrates both the elegance and utility of classical probability distributions. The problem, rooted in the Negative Binomial distribution, offers a clear framework for calculating the likelihood of specific stopping times and understanding the stochastic nature of repeated independent trials.

Key takeaways include:

- The probability that the sixth 5 occurs exactly at the \(n^{th}\) roll is given by the Negative Binomial formula.
- The expected number of rolls to achieve six 5's is 36.
- The distribution of total trials needed is discrete, with probabilities that can be explicitly computed.

Understanding such probability models enhances our grasp of random processes and equips us with tools to analyze complex, real-world scenarios where outcomes are governed by chance. Whether in gaming, manufacturing, or scientific research, these principles remain foundational for interpreting and predicting stochastic phenomena.

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In essence, the experiment of rolling a die until six 5s appear is a rich illustration of the interplay between probability distributions and stochastic processes, providing insights that extend well beyond the simple act of rolling dice.

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