Compute The Given Integral By First Identifying The Integral As The Volume Of A Solid. V - 16 X2 - Y2

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In calculus, many integrals can be interpreted geometrically, providing a more intuitive understanding of the problem and often simplifying the computation process. One powerful approach involves recognizing the integral as the volume of a solid bounded by specific surfaces. In this article, we will explore how to evaluate the integral representing the volume of the solid defined by the surface \(V = 16x^2 + y^2 \). This method not only enhances conceptual clarity but also leverages the symmetry and properties of the surfaces involved to facilitate the calculation.

Understanding the Geometric Interpretation of the Integral

What Does the Integral Represent?

The given integral can be viewed as the volume enclosed between a surface \(V = 16x^2 + y^2 \) and a reference plane (often the \(xy \)-plane or another plane depending on the problem context). Recognizing it as a volume allows us to:

- Visualize the problem in three dimensions.
- Use geometric methods such as setting up limits that correspond to the shape's boundaries.
- Simplify the integration process by exploiting symmetry and known volume formulas.

Why Geometric Interpretation Is Helpful

Understanding the integral as a volume:

- Clarifies the shape we're integrating over.
- Helps in choosing the appropriate coordinate system.
- Makes it easier to determine bounds of integration.
- Provides insight into the nature of the surface, such as whether it forms an ellipsoid, paraboloid, or other surfaces.

Identifying the Surface and Boundaries

Analyzing the Surface Equation $V = 16x^2 + y^2$

The surface can be rewritten as:

$$
z = 16x^2 + y^2
$$

This is a paraboloid opening upward. It is symmetric with respect to the z-axis and has an elliptical cross-section when intersected with planes parallel to the xy-plane.

Determining the Region of Integration

Typically, the integral's limits relate to the projection of the volume onto the xy-plane. To evaluate the volume, we need to:

- Find the projection region R in the xy-plane.
- Express the volume as:

$$
V = \iint_{R} z \, dA
$$

where $z = 16x^2 + y^2$.

Alternatively, if the problem specifies bounds for z, we would set limits accordingly.

Choosing an Appropriate Coordinate System

Why Switch to Cylindrical or Elliptical Coordinates?

Given the shape involves quadratic forms like $16x^2 + y^2$, switching to coordinate systems that simplify these expressions can make integration more straightforward.

- Cylindrical Coordinates: Useful if the surface has rotational symmetry.
- Elliptical Coordinates: Useful when the surface involves elliptical cross-sections.

Transforming to Elliptical Coordinates

Because the surface involves $16x^2 + y^2$, it’s natural to consider elliptical coordinates defined as:

$$
x = \frac{r}{a} \cos \theta, \quad y = r \sin \theta
$$

where a is a scaling factor aligning with the coefficients in the quadratic form.

Alternatively, consider the substitution:

$$
u = 4x, \quad v = y
$$

which simplifies the surface to:

$$
z = u^2 + v^2
$$

This substitution transforms the problem into evaluating the volume under the paraboloid $z = u^2 + v^2$, with the projection region scaled accordingly.

Setting Up the Integral for Volume Calculation

Expressing the Volume as a Double Integral

Given the surface $z = 16x^2 + y^2$, the volume over a region R in the xy-plane can be written as:

$$
V = \iint_{R} (16x^2 + y^2) \, dx\,dy
$$

Alternatively, if the problem specifies bounds for z, such as z from 0 to some function, then the integral becomes:

$$
V = \iint_{R} \left(\text{upper surface} - \text{lower surface} \right) \, dA
$$

Assuming the volume is bounded above by the surface $z = 16x^2 + y^2$ and below by the xy-plane $z=0$, the limits are determined by where the surface intersects the plane.

Determining the Projection Region R

To find the projection region, set $z=0$:

$$
0 = 16x^2 + y^2
$$

which implies:

$$
x=0, \quad y=0
$$

This indicates the surface intersects the xy-plane at the origin, and the region of interest could be a circular or elliptical region depending on the bounds imposed.

If the problem defines a specific boundary such as $z = c$ for some constant c, then:

$$
c = 16x^2 + y^2
$$

which describes an elliptical boundary:

$$
\frac{x^2}{(c/16)} + \frac{y^2}{c} = 1
$$

Thus, the projection region R is the interior of this ellipse.

Calculating the Volume Using Coordinate Transformation

Transforming the Region into a Circle

Using the substitution:

$$
u = 4x \quad \Rightarrow \quad x=\frac{u}{4}
$$
$$
v = y
$$

then,

$$
z = u^2 + v^2
$$

and the region R in the xy-plane becomes:

$$
\frac{u^2}{c} + \frac{v^2}{c} \leq 1
$$

which simplifies to:

$$
u^2 + v^2 \leq c
$$

This is a circle of radius $\sqrt{c}$ in the uv-plane.

Expressing the Volume Integral in uv-Coordinates

The Jacobian of the transformation is:

$$
\left| \frac{\partial (x,y)}{\partial (u,v)} \right| = \frac{1}{4}
$$

since:

$$
dx\,dy = \frac{1}{4} du\,dv
$$

The volume becomes:

$$
V = \iint_{u^2+v^2 \leq c} (u^2 + v^2) \times \frac{1}{4} du\,dv
$$

Switching to polar coordinates:

$$
u = r \cos \theta, \quad v = r \sin \theta
$$

where $r \in [0, \sqrt{c}]$, $\theta \in [0, 2\pi]$, and the Jacobian is r.

The integral simplifies to:

$$
V = \frac{1}{4} \int_0^{2\pi} \int_0^{\sqrt{c}} r^2 \times r\, dr\, d\theta = \frac{1}{4} \int_0^{2\pi} \int_0^{\sqrt{c}} r^3\, dr\, d\theta
$$

Calculating:

$$
V = \frac{1}{4} \int_0^{2\pi} \left[\frac{r^4}{4} \right]_0^{\sqrt{c}} d\theta = \frac{1}{4} \int_0^{2\pi} \frac{(\sqrt{c})^4}{4} d\theta
$$

which simplifies to:

$$
V = \frac{1}{4} \times \frac{c^2}{4} \times \int_0^{2\pi} d\theta = \frac{c^2}{16} \times 2\pi
$$

$$
V = \frac{\pi c^2}{8}
$$

Thus, the volume enclosed beneath the surface $z = 16x^2 + y^2$ up to height c is:

$$
\boxed{
V = \frac{\pi c^2}{8}
}
$$

Summary and Application

Key Steps to Compute the Volume

1. Identify the surface and interpret the integral as the volume under that surface.
2. Determine the projection region in the xy-plane, often by setting the surface equal to a constant or zero.
3. Choose an appropriate coordinate system (Cartesian, cylindrical, elliptical, or polar) to simplify the integral.
4. Transform the integral using substitution and compute the Jacobian as needed.
5. Set up the double integral with correct bounds based on the

Frequently Asked Questions

How can the integral $V = 16x^2 - y^2$ be interpreted as the volume of a solid?

The integral can be viewed as representing the volume under a surface defined by the function $z = 16x^2 - y^2$ over a specific region in the xy-plane. By identifying this as a volume, we can set up a double integral over that region to compute the total volume.

What is the first step in computing the volume from the given integral $V = 16x^2 - y^2$?

The first step is to interpret the integral as a volume by recognizing the surface $z = 16x^2 - y^2$ and determining the region in the xy-plane over which to integrate, typically where the surface is above or below the xy-plane depending on the problem.

How do you find the boundary region for integrating the volume of the solid?

You find the boundary by setting the surface equal to zero ($16x^2 - y^2 = 0$) to determine where the surface intersects the xy-plane, which helps define the limits for the double integral. This gives $y = \pm 4x$, forming the boundary region.

What coordinate system is most convenient for integrating the volume of this solid?

Using Cartesian coordinates is straightforward here, but for certain regions, transforming to polar or other coordinate systems can simplify the integration, especially if the boundary region or the integrand is symmetric.

How does symmetry help in computing the volume of this solid?

Since the surface is symmetric about the x-axis (because the integrand involves y^2), we can compute the volume over one symmetric region and then multiply by 2 to find the total volume, simplifying calculations.

What is the significance of identifying the integral as a volume when solving for the value of V?

Identifying the integral as a volume helps in choosing the appropriate limits of integration and understanding the geometric interpretation, making it easier to set up and evaluate the integral accurately.

Can you outline the steps to compute the volume given $V = 16x^2 - y^2$?

Yes. First, find the boundary region where the surface intersects the xy-plane by setting $16x^2 - y^2 = 0$. Next, determine the limits for x and y based on this boundary. Then, set up the double integral of $z = 16x^2 - y^2$ over this region. Finally, evaluate the integral to find the volume.

What are common challenges when interpreting an integral as a volume and how can they be addressed?

Common challenges include correctly identifying the boundary region and limits of integration. These can be addressed by analyzing the surface equation, setting it equal to zero to find intersections, and carefully sketching the boundary region before integrating.

Additional Resources

Compute the Given Integral By First Identifying The Integral As The Volume Of A Solid

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Introduction

Calculating integrals often involves more than straightforward antiderivatives; it can also be approached through geometric interpretations. One of the most elegant and insightful methods is recognizing certain integrals as volumes of three-dimensional solids. This approach not only simplifies the calculation but also offers a deeper understanding of the problem's spatial structure.

In this detailed exploration, we focus on the integral associated with the volume of a solid defined by the surface $V = 16x^2 + y^2$. The goal is to compute the integral by first interpreting it as the volume of this solid, which leads us into the realm of double integrals, coordinate transformations, and geometric intuition.

Understanding the Problem: The Role of the Surface $V = 16x^2 + y^2$

Before delving into calculations, it is crucial to understand the surface and its geometric significance.

The Surface Equation

- The surface is given as:

$$
V = 16x^2 + y^2
$$

- This is a paraboloid, opening upward along the V-axis, with its minimum at the origin $(0,0)$.

- In the context of volume calculations, this surface often forms the boundary of a solid region when combined with certain limits or other surfaces.

Visualizing the Surface

- The paraboloid is symmetric about both the x- and y-axes due to the quadratic terms.

- The cross-sections perpendicular to the V-axis are ellipses, with axes scaled differently in the x and y directions.

- Given the structure, the volume enclosed beneath or within certain bounds of this surface can be interpreted as the volume of a solid.

Interpreting the Integral as a Volume

The Concept of Volume via Double Integrals

- A double integral over a region R in the xy-plane can be interpreted as the volume under a surface $z = f(x,y)$:

$$
\text{Volume} = \iint_{R} f(x,y) \, dx\,dy
$$

- If the function $f(x,y)$ describes the height of the surface above the xy-plane, integrating over the region R gives the volume of the solid bounded above by the surface.

Recognizing the Given Surface as a Boundary

- In this scenario, the surface $V = 16x^2 + y^2$ can be viewed as a height function $z = 16x^2 + y^2$.

- To compute a volume associated with this surface, one often considers a specific region R in the xy-plane, such as the projection of the solid onto the xy-plane, bounded by a certain curve or inequality.

- For example, the volume under the surface above a region R is:

$$
\text{Volume} = \iint_{R} (16x^2 + y^2) \, dx\,dy
$$

Determining the Region R

The key to computing the volume via the integral is identifying the region R in the xy-plane over which to integrate.

Typical Boundaries

- The problem often involves a boundary condition, such as:

$$
V \leq \text{constant}
$$

- For instance, integrating over the region where:

$$
V = 16x^2 + y^2 \leq c
$$

- This inequality describes a filled region under the paraboloid up to height c.

Example: The Region $V \leq 16$

Suppose the problem involves calculating the volume under the surface where:

$$
V = 16x^2 + y^2 \leq 16
$$

- This inequality describes a closed region in the xy-plane:

$$
16x^2 + y^2 \leq 16
$$

- Dividing both sides by 16:

$$
x^2 + \frac{y^2}{16} \leq 1
$$

- This is an ellipse centered at the origin with semi-axes:

- $a = 1$ in the x-direction,

- $b = 4$ in the y-direction.

Thus, the region R is the interior of this ellipse:

$$
x^2 + \frac{y^2}{16} \leq 1

\]

Setting Up the Double Integral

Integral Expression

- The volume V under the surface over the elliptical region is:

$$
V = \iint_{R} (16x^2 + y^2) \, dx\,dy
$$

- Since R is the ellipse $x^2 + y^2/16 \leq 1$, the limits are defined by this inequality.

Coordinate Transformation for Simplification

- To evaluate this integral efficiently, it's beneficial to transform the elliptical region into a circle via substitution.

- Transform:

$$
x = r \cos \theta
$$
$$
y = 4 r \sin \theta
$$

- The Jacobian determinant of this transformation is:

$$
J = \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| = \left|
\begin{array}{cc}
\cos \theta & -r \sin \theta \\
4 \sin \theta & 4 r \cos \theta
\end{array}
\right| = 4 r (\cos^2 \theta + \sin^2 \theta) = 4 r
$$

- Limits:

- r ranges from 0 to 1,

- θ ranges from 0 to 2π.

Evaluating the Integral

Transform the integrand

Express $16x^2 + y^2$ in terms of r and θ:

$$
16x^2 + y^2 = 16 (r^2 \cos^2 \theta) + (4 r \sin \theta)^2 = 16 r^2 \cos^2 \theta + 16 r^2 \sin^2 \theta = 16 r^2 (\cos^2 \theta + \sin^2 \theta) = 16 r^2
$$

Write the integral

$$
V = \int_{0}^{2\pi} \int_{0}^{1} 16 r^2 \times 4 r \, dr\, d\theta
$$

where the Jacobian $J = 4 r$ is included in the differential area element:

$$
dx\,dy = J dr\, d\theta
$$

Thus:

$$
V = \int_{0}^{2\pi} \int_{0}^{1} 16 r^2 \times 4 r \, dr\, d\theta = \int_{0}^{2\pi} \int_{0}^{1} 64 r^3 \, dr\, d\theta
$$

Computing the Integral

1. Inner integral over r:

$$
\int_{0}^{1} 64 r^3 \, dr = 64 \times \frac{r^4}{4} \bigg|_{0}^{1} = 64 \times \frac{1}{4} = 16
$$

2. Outer integral over θ:

$$
\int_{0}^{2\pi} d\theta = 2\pi
$$

3. Final volume:

$$
V = 16 \times 2\pi = 32 \pi
$$

Geometric Interpretation and Final Result

The computed volume $V = 32 \pi$ corresponds to the volume beneath the surface $V = 16x^2 + y^2$ over the elliptical region $x^2 + y^2/16 \leq 1$.

This approach exemplifies how recognizing the integral as a volume simplifies the calculation:

- Step 1: Identify the boundary of the region in the xy-plane.

- Step 2: Recognize the surface as a height function.

- Step 3: Transform the region into a circle using substitution to leverage symmetry.

- Step 4: Set up and evaluate the double integral in the transformed coordinates.

Broader Applications and Additional Insights

When to Use Geometric Interpretation

- Recognizing an integral as a volume is particularly useful when the integrand or the region has geometric symmetry.

- It allows the use of coordinate transformations (polar, elliptical, cylindrical) to simplify the integral.

- This method also provides visual insights into the problem, making complex integrations manageable.

Variations and Generalizations

- If the boundary condition or the limits change, the same principles apply—identify the region, transform as needed, and compute the volume.

- For more complex surfaces or regions, advanced techniques such as multiple coordinate transformations or leveraging symmetry may be necessary.

- The approach extends naturally to other solids like paraboloids, hyperboloids, and more, with appropriate transformations.

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