Consider A Discrete-time LTI System With Impulse Response H[n] = (1/2)^n U[n]. Use Fourier Transforms

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Introduction

In the study of discrete-time linear time-invariant (LTI) systems, the impulse response $H[n]$ plays a fundamental role in characterizing the system's behavior. When analyzing such systems, Fourier transforms serve as powerful tools, transforming the time domain signals into the frequency domain, thereby simplifying the convolution process and enabling a deeper understanding of the system's spectral properties. In this article, we explore the specific case where the impulse response is given by $H[n] = \left(\frac{1}{2}\right)^n U[n]$, with $U[n]$ being the unit step function. We will analyze this system using Fourier transforms, derive its frequency response, and interpret the results in terms of system behavior and stability.

Understanding the Impulse Response $H[n]$

Definition and Properties of $H[n]$

The impulse response of the system is given as:

$$
H[n] = \left(\frac{1}{2}\right)^n U[n]
$$

where:

- $\left(\frac{1}{2}\right)^n$ is an exponentially decaying sequence.

- $U[n]$ is the unit step function, defined as:

$$
U[n] = \begin{cases}
1 & n \ge 0 \\
0 & n < 0
\end{cases}
$$

This ensures $H[n]$ is causal, meaning the system's output at time n depends only on current and past inputs.

Key properties:

- Causality: Due to the multiplication by $U[n]$, the impulse response is zero for $n < 0$.
- Stability: The system's BIBO (Bounded Input Bounded Output) stability depends on the absolute integrability of the impulse response.
- Decay rate: The exponential decay factor $(1/2)^n$ ensures the impulse response diminishes as $n \to \infty$.

Fourier Transform of $H[n]$

Definition of the Discrete-Time Fourier Transform (DTFT)

The DTFT of a sequence $h[n]$ is defined as:

$$
H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j \omega n}
$$

For the given $H[n]$, since $H[n] = 0$ for $n < 0$, the sum reduces to:

$$
H(e^{j\omega}) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j \omega n}
$$

Calculating the Frequency Response

This is a geometric series with ratio:

$$
r = \left(\frac{1}{2}\right) e^{-j \omega}
$$

The sum of an infinite geometric series, provided $|r| < 1$, is:

$$
\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}
$$

Since $|r| = \left| \frac{1}{2} e^{-j \omega} \right| = \frac{1}{2} < 1$, the series converges, and the frequency response is:

$$
H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} e^{-j \omega}}
$$

Expressing $H(e^{j\omega})$ in a More Useful Form

Multiplying numerator and denominator by the complex conjugate of the denominator's denominator:

$$
H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} e^{-j \omega}} \times \frac{1 - \frac{1}{2} e^{j \omega}}{1 - \frac{1}{2} e^{j \omega}} = \frac{1 - \frac{1}{2} e^{j \omega}}{\left| 1 - \frac{1}{2} e^{-j \omega} \right|^2}
$$

Calculating the denominator modulus squared:

$$
\left| 1 - \frac{1}{2} e^{-j \omega} \right|^2 = \left(1 - \frac{1}{2} e^{-j \omega} \right) \left(1 - \frac{1}{2} e^{j \omega} \right)
$$

$$
= 1 - \frac{1}{2} e^{j \omega} - \frac{1}{2} e^{-j \omega} + \frac{1}{4}
$$

\]

Recognizing that $e^{j \omega} + e^{-j \omega} = 2 \cos \omega$:

$$
= 1 - \frac{1}{2} \times 2 \cos \omega + \frac{1}{4} = 1 - \cos \omega + \frac{1}{4}
$$

$$
= \frac{5}{4} - \cos \omega
$$

Thus, the frequency response simplifies to:

$$
H(e^{j \omega}) = \frac{1 - \frac{1}{2} e^{j \omega}}{\frac{5}{4} - \cos \omega}
$$

Analysis of the Frequency Response

Magnitude and Phase Response

The magnitude response is:

$$
|H(e^{j \omega})| = \frac{\left| 1 - \frac{1}{2} e^{j \omega} \right|}{\sqrt{\left(\frac{5}{4} - \cos \omega \right)^2}}
$$

Calculating numerator magnitude:

$$
\left| 1 - \frac{1}{2} e^{j \omega} \right| = \sqrt{\left(1 - \frac{1}{2} \cos \omega \right)^2 + \left(- \frac{1}{2} \sin \omega \right)^2}
$$

$$
= \sqrt{\left(1 - \frac{1}{2} \cos \omega \right)^2 + \frac{1}{4} \sin^2 \omega}
$$

\]

The phase response, $\phi(\omega) = \arg H(e^{j \omega})$, can be obtained by considering the arguments of numerator and denominator.

Stability and System Behavior

Since $H[n]$ is absolutely summable:

$$
\sum_{n=0}^{\infty} \left| \left(\frac{1}{2}\right)^n \right| = \frac{1}{1 - \frac{1}{2}} = 2
$$

the system is BIBO stable.

The exponential decay in $H[n]$ indicates that the system acts as a low-pass filter, attenuating high-frequency components. The spectral analysis reveals that the system's magnitude response peaks at $\omega = 0$, indicating maximum gain at DC (zero frequency), and diminishes as ω approaches π.

Interpretation and Applications

Physical Meaning of the Frequency Response

The frequency response $H(e^{j \omega})$ characterizes how different frequency components of an input signal are modified as they pass through the system. The causality and exponential decay of $H[n]$ imply that the system responds smoothly to inputs, emphasizing low frequencies.

- Low-pass filtering: The system's ability to pass low-frequency signals with minimal attenuation while attenuating high-frequency signals makes it suitable for applications like smoothing and noise reduction.

- Time-domain implications: The causal exponential impulse response signifies that the system's output is a weighted sum of past inputs, with weights decreasing exponentially over time.

Practical Examples

- Signal smoothing: This system can be used to smooth noisy signals, reducing the impact of rapid, high-frequency fluctuations.

- System modeling: The exponential impulse response models physical systems with damping characteristics, such as RC circuits or mechanical systems with friction.

- Filter design: Understanding the frequency response aids in designing filters with desired spectral properties.

Conclusion

Analyzing the discrete-time LTI system with impulse response $H[n] = \left(\frac{1}{2}\right)^n U[n]$ reveals rich spectral properties that are fundamental in signal processing. Using Fourier transforms, we derived the system's frequency response:

$$
H(e^{j \omega}) = \frac{1 - \frac{1}{2} e^{j \omega}}{\frac{5}{4} - \cos \omega}
$$

This response demonstrates low-pass behavior, causality, and stability, making such systems vital in applications requiring smoothing or filtering. Fourier analysis provides a comprehensive understanding of how the system modifies signals in the frequency domain, guiding design and

Frequently Asked Questions

What is the impulse response H[n] of the given discrete-time LTI system?

The impulse response is $H[n] = (1/2)^n U[n]$, where $U[n]$ is the unit step function, meaning $H[n] = (1/2)^n$ for $n \geq 0$ and 0 otherwise.

How do you compute the Fourier Transform of the impulse response $H[n] = (1/2)^n U[n]$?

The Fourier Transform $H(e^{j\omega})$ is computed as the sum from n=0 to ∞ of $H[n] e^{-j\omega n} = \sum_{n=0}^\infty (1/2)^n e^{-j\omega n}$, which is a geometric series.

What is the frequency response $H(e^{j\omega})$ of the system?

The frequency response is $H(e^{j\omega}) = 1 / (1 - (1/2) e^{-j\omega})$, valid for $|(1/2) e^{-j\omega}| < 1$, which simplifies to $H(e^{j\omega}) = 1 / (1 - (1/2) e^{-j\omega})$.

Is the Fourier Transform of H[n] well-defined for all ω?

Yes, the Fourier Transform converges for all ω because the geometric series converges when $|(1/2) e^{-j\omega}| = 1/2 < 1$.

What is the magnitude response $|H(e^{j\omega})|$ of the system?

The magnitude response is $|H(e^{j\omega})| = 1 / \sqrt{1 - (1/2)(e^{j\omega} + e^{-j\omega}) + (1/4)}$, which simplifies to $|H(e^{j\omega})| = 1 / \sqrt{1 - (1/2) 2 \cos \omega + 1/4} = 1 / \sqrt{(3/4) - \cos \omega}$.

How does the impulse response H[n] relate to the system's stability in the frequency domain?

Since the Fourier Transform $H(e^{j\omega})$ is bounded and well-defined, the system is BIBO (Bounded Input Bounded Output) stable, as the impulse response is absolutely summable.

Can you express $H(e^{j\omega})$ in a rational form for analysis?

Yes, $H(e^{j\omega}) = 1 / (1 - (1/2) e^{-j\omega})$, which is a rational function in $e^{j\omega}$ and can be analyzed for pole-zero locations.

What is the significance of the pole of $H(e^{j\omega})$ in the z-plane?

The pole is at $z = (1/2)$, located inside the unit circle, indicating that the system is causal and stable, with a pole at $z = (1/2)$.

How can the inverse Fourier Transform be used to recover H[n]?

Given $H(e^{j\omega})$, the inverse Fourier Transform involves integrating over ω: $H[n] = (1/2\pi) \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$, which recovers the original impulse response.

Additional Resources

Fourier Analysis of a Discrete-Time LTI System with Impulse Response H[n] = (1/2)^n U[n]

Introduction

In the study of discrete-time linear time-invariant (LTI) systems, understanding the relationship between the system's impulse response and its frequency response is fundamental. Fourier transforms serve as powerful tools to analyze and characterize these systems, facilitating insights into their behavior in the frequency domain. The specific system we examine here is defined by its impulse response:

$$
H[n] = \left(\frac{1}{2} \right)^n U[n]
$$

where $U[n]$ is the unit step function ensuring causality, i.e., $H[n] = 0$ for $n < 0$. This particular impulse response describes a simple yet insightful exponential decay process starting at $n=0$. Our goal is to explore this system thoroughly using Fourier analysis, deriving the frequency response, discussing its properties, and interpreting its implications.

Understanding the System's Impulse Response

Definition and Characteristics

- Causality: Because of the unit step function $U[n]$, the system is causal, meaning it responds only to current and past inputs.
- Exponential Decay: The impulse response exhibits exponential decay with ratio $1/2$, indicating that the system filters signals by damping higher frequencies.
- Finite Energy: Since $|H[n]| = (1/2)^n$ for $n \geq 0$, the sequence is absolutely summable, confirming that it is a finite-energy system.

Significance in Signal Processing

This type of impulse response models many real-world systems like RC circuits, certain filters, or decay processes in digital signal processing. Understanding its frequency response helps in designing systems that can either pass or attenuate specific frequency components.

Fourier Transform of the Impulse Response

Discrete-Time Fourier Transform (DTFT)

The primary tool for analyzing $H[n]$ in the frequency domain is the DTFT, defined as:

$$
H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} H[n] e^{-j\omega n}
$$

Given $H[n] = 0$ for $n < 0$, the sum reduces to:

$$
H(e^{j\omega}) = \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n e^{-j\omega n}
$$

This is a geometric series in n, which converges absolutely because $|(1/2) e^{-j\omega}| = 1/2 < 1$.

Derivation of the Frequency Response

The sum of a geometric series:

$$
\sum_{n=0}^\infty r^n = \frac{1}{1 - r}, \quad \text{for} \ |r| < 1
$$

Applying this to our sum:

$$
H(e^{j\omega}) = \frac{1}{1 - \left(\frac{1}{2} e^{-j\omega} \right)} \quad \text{for} \ |e^{-j\omega}/2| < 1
$$

Since $|e^{-j\omega}| = 1$, the convergence condition simplifies to:

$$
\left| \frac{1}{2} \right| < 1
$$

which always holds, confirming the sum converges for all ω.

Thus, the frequency response is:

\[

\boxed{
H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}
}
\]

Analysis of the Frequency Response

Magnitude Response

The magnitude of \(H(e^{j\omega}) \) is:

\[
|H(e^{j\omega})| = \left| \frac{1}{1 - \frac{1}{2} e^{-j\omega}} \right| = \frac{1}{\left| 1 - \frac{1}{2} e^{-j\omega} \right|}
\]

Expressing the denominator:

\[
\left| 1 - \frac{1}{2} e^{-j\omega} \right| = \sqrt{ \left(1 - \frac{1}{2} \cos \omega \right)^2 + \left(\frac{1}{2} \sin \omega \right)^2 }
\]

which simplifies to:

\[
|H(e^{j\omega})| = \frac{1}{\sqrt{ \left(1 - \frac{1}{2} \cos \omega \right)^2 + \frac{1}{4} \sin^2 \omega }}
\]

Expanding:

\[
|H(e^{j\omega})| = \frac{1}{\sqrt{ 1 - \cos \omega + \frac{1}{4} \cos^2 \omega + \frac{1}{4} \sin^2 \omega }}
\]

Using the Pythagorean identity \(\cos^2 \omega + \sin^2 \omega = 1 \):

\[
|H(e^{j\omega})| = \frac{1}{\sqrt{ 1 - \frac{1}{2} \cos \omega + \frac{1}{4} }}
\]

which simplifies to:

$$
|H(e^{j\omega})| = \frac{1}{\sqrt{ \frac{5}{4} - \frac{1}{2} \cos \omega }}
$$

This expression determines how the system scales different frequency components. Notably:

- The magnitude peaks at $\omega = 0$, where $\cos 0 = 1$:

$$
|H(e^{j0})| = \frac{1}{\sqrt{\frac{5}{4} - \frac{1}{2} \times 1 }} = \frac{1}{\sqrt{\frac{5}{4} - \frac{1}{2}}} = \frac{1}{\sqrt{\frac{5}{4} - \frac{2}{4}}} = \frac{1}{\sqrt{\frac{3}{4}}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}
$$

- The minimum occurs at $\omega = \pi$, where $\cos \pi = -1$:

$$
|H(e^{j\pi})| = \frac{1}{\sqrt{\frac{5}{4} - \frac{1}{2} \times (-1)}} = \frac{1}{\sqrt{\frac{5}{4} + \frac{1}{2}}} = \frac{1}{\sqrt{\frac{5}{4} + \frac{2}{4}}} = \frac{1}{\sqrt{\frac{7}{4}}} = \frac{2}{\sqrt{7}}
$$

This indicates a low-pass characteristic, as the system passes low frequencies (near $\omega=0$) with higher gain and attenuates high frequencies (near $\omega=\pi$).

Phase Response

The phase of $H(e^{j\omega})$ is obtained from:

$$
\angle H(e^{j\omega}) = - \arg \left(1 - \frac{1}{2} e^{-j\omega} \right)
$$

Expressing the denominator in rectangular form:

$$
1 - \frac{1}{2} e^{-j\omega} = 1 - \frac{1}{2} (\cos \omega - j \sin \omega) = \left(1 - \frac{1}{2} \cos \omega \right) + j \frac{1}{2} \sin \omega
$$

The phase is:

$$
\angle H(e^{j\omega}) = - \arctan \left(\frac{\frac{1}{2} \sin \omega}{1 - \frac{1}{2} \cos \omega} \right)
$$

which gives insight into the phase shift introduced at different frequencies.

System Properties and Implications

Stability

Since the impulse response is absolutely summable:

$$
\sum_{n=0}^\infty \left| \left(\frac{1}{2} \right)^n \right| = \frac{1}{1 - \frac{1}{2}} = 2 < \infty
$$

the system is BIBO (Bounded Input, Bounded Output) stable. This is a fundamental requirement in practical systems.

Causality and Realizability

The impulse response is causal and right-sided, making the system physically realizable and suitable for digital implementation.

Filter Type

The frequency response reveals that the system behaves as a low-pass filter, attenuating high-frequency signals while passing low-frequency components with relatively higher gain.

Effect on Signals

- Smoothing and Damping: Signals passing through this system will experience exponential decay, especially at higher frequencies.
- Phase Shift: The phase response indicates a frequency-dependent shift, which can cause distortion in time-domain signals if not compensated.

Consider A Discrete Time Lti System With Impulse Response

H N 1 2 N U N Use Fourier Transforms

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