Consider The Bases B = {u1, U2} And B' = {u'1, U'2} For R2, Where U1 = [2, 2], U2 = [4, -1], [u1 = 1,

Consider The Bases B = {u1, U2} And B' = {u'1, U'2} For R2, Where U1 = [2, 2], U2 = [4, -1], [u1 = 1, and continue naturally.

Understanding the concepts of bases in R^2 is fundamental to linear algebra, as it allows us to analyze the structure of vector spaces, change coordinate systems, and solve systems of equations efficiently. In this article, we will explore the properties of different bases, how to express vectors relative to these bases, and the significance of basis transformations. Our focus is on the specific bases B and B' defined by the vectors U1, U2, and their primed counterparts, providing a detailed walkthrough of their characteristics and applications.

What Are Bases in R^2?

Definition of a Basis

In the context of R^2, a basis is a set of two vectors that are linearly independent and span the entire space. This means any vector in R^2 can be represented as a unique linear combination of the basis vectors. Formally, a set $\{v_1, v_2\}$ is a basis for R^2 if:

- v_1 and v_2 are linearly independent (neither is a scalar multiple of the other)
- Any vector v in R^2 can be written as $v = av_1 + bv_2$ for some scalars a and b

Importance of Choosing a Basis

Choosing an appropriate basis simplifies calculations, especially when transforming coordinates or solving systems. Different bases can reveal different properties of vectors, such as their orientation relative to certain axes or subspaces.

Examining the Given Bases

Base B = {u1, U2}

Given:
- U1 = [2, 2]
- U2 = [4, -1]

Note that the notation indicates these are vectors in R^2:
- u1 = [2, 2]
- U2 = [4, -1]

The basis B is therefore composed of these two vectors.

Base B' = {u'1, U'2}

While the explicit vectors for B' are not fully specified in the initial statement, typically, they would be another set of two vectors in R^2, possibly related to B via a linear transformation or change of basis. Understanding how to transition between B and B' will be crucial for many applications.

Verifying if the Vectors Form a Valid Basis

Check for Linear Independence

To confirm whether {u1, U2} forms a basis, we need to verify that these vectors are linearly independent. This involves checking if the only solution to the equation:

a [2, 2] + b [4, -1] = [0, 0]

is a = 0 and b = 0.

Calculations:
- Set up the equations:
- 2a + 4b = 0
- 2a - b = 0

From the second:
- 2a - b = 0 → b = 2a

Substitute into the first:
- 2a + 4(2a) = 0
- 2a + 8a = 0
- 10a = 0 → a = 0

Then, b = 2a = 0

Since the only solution is a = 0, b = 0, the vectors are linearly independent, and thus form a basis.

Expressing Vectors in Different Bases

Coordinate Representation in Basis B

Any vector $v = [x, y]$ in R^2 can be written as a linear combination of the basis vectors:

$v = \alpha u_1 + \beta U_2$

This translates to the system:
- $2\alpha + 4\beta = x$
- $2\alpha - \beta = y$

To find α and β:
- From the second:
- $2\alpha - \beta = y \rightarrow \beta = 2\alpha - y$
- Substitute into the first:
- $2\alpha + 4(2\alpha - y) = x$
- $2\alpha + 8\alpha - 4y = x$
- $10\alpha = x + 4y$
- $\alpha = (x + 4y) / 10$

Then:
- $\beta = 2 [(x + 4y)/10] - y = (2x + 8y)/10 - y = (2x + 8y - 10y)/10 = (2x - 2y)/10 = (x - y)/5$

Thus, the coordinates of v in basis B are:

$[\alpha, \beta] =$
$$
\left[\frac{x + 4y}{10}, \frac{x - y}{5}\right]
$$

Coordinate Representation in Basis B'

Similarly, to express v in basis B', we need the vectors u'_1 and U'_2. If their explicit coordinates are provided, the process mirrors the one above: setting up the linear system and solving for the scalar coefficients.

Change of Basis and Transformation Matrices

Constructing the Basis Change Matrix

The matrix that converts coordinates from basis B to the standard basis (the usual x-y axes) is:

$$ P_B = \begin{bmatrix} 2 & 4 \\ 2 & -1 \end{bmatrix} $$

Similarly, to convert from the standard basis back to B, we need the inverse of P_B:

$$ P_B^{-1} $$

Calculating the inverse involves:
- Computing the determinant:
- $\det(P_B) = (2)(-1) - (4)(2) = -2 - 8 = -10$
- The inverse:

$$ P_B^{-1} = \frac{1}{-10} \begin{bmatrix} -1 & -4 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.4 \\ 0.2 & -0.2 \end{bmatrix} $$

This inverse matrix allows us to convert any vector from standard coordinates to basis B coordinates.

Implications for Linear Algebra Applications

The ability to switch between bases is fundamental in:

- Diagonalizing matrices
- Solving differential equations
- Performing coordinate transformations in computer graphics
- Analyzing vector projections and decompositions

Practical Applications of Basis Transformations

Coordinate Changes in Engineering and Physics

In engineering, changing the basis can simplify complex systems, making analysis more manageable. For example, in mechanics, choosing a basis aligned with a particular force vector can streamline calculations.

In physics, basis transformations are critical when shifting between different reference frames, such as from Cartesian to polar coordinates, or when analyzing vector fields.

Data Representation and Machine Learning

In data science, representing data in different bases can reveal underlying structures. Principal Component Analysis (PCA), for instance, finds a basis that maximizes variance, improving data interpretability.

Summary and Key Takeaways

- A basis in R^2 consists of two linearly independent vectors that span the entire space.
- Given vectors U1 = [2, 2] and U2 = [4, -1], we verified their independence and their capacity to form a basis.
- Expressing vectors in different bases involves solving linear systems, which can be facilitated by basis change matrices.
- Understanding bases and transformations is essential for advanced applications across science and engineering.
- Changing bases allows for simplified computations, better data interpretation, and more flexible problem-solving strategies.

In conclusion, the exploration of bases B and B' in R^2 illuminates the importance of linear independence, coordinate transformations, and the practical utility of basis change matrices. Mastery of these concepts provides a powerful toolkit for tackling a wide array of problems in mathematics, physics, engineering, and beyond.

Frequently Asked Questions

How do you find the change of basis matrix from B to B' in R^2?

To find the change of basis matrix from B to B', you express each vector of B in terms of B' coordinates or vice versa. Specifically, you can write the vectors of B in the basis B' and arrange these as columns to form the matrix.

What are the coordinates of u1 = [2, 2] in basis B' = {u'1, u'2}?

To find the coordinates of u1 in B', solve the system u1 = a u'1 + b u'2. Similarly for u2, you do the same. You need to know the vectors u'1 and u'2 to proceed with this calculation.

Given u1 = [2, 2] and u2 = [4, -1], how can we verify if B = {u1, u2} is a basis for R^2?

Check if u1 and u2 are linearly independent by computing their determinant or checking if one is a scalar multiple of the other. If they are linearly independent, then B is a basis for R^2.

How do you compute the coordinates of a vector in basis B?

To compute the coordinates of a vector v in basis B = {u1, u2}, solve the linear system v = a u1 + b u2 for a and b. The solutions (a, b) are the coordinates of v in B.

What is the significance of the bases B and B' in understanding transformations in R^2?

The bases B and B' allow us to represent vectors and linear transformations in different coordinate systems, facilitating easier computations, especially when transforming vectors between different bases.

How can you find the transition matrix from basis B to basis B' given their vectors?

Construct matrices with the basis vectors of B and B' as columns. The transition matrix from B to B' is obtained by expressing each vector of B in terms of B' and assembling these as columns. Alternatively, it can be found by multiplying the inverse of the B' matrix by the B matrix.

Why is it important to verify that the basis vectors are linearly independent when working with bases?

Because only linearly independent vectors form a basis, ensuring that every vector in the space can be uniquely expressed as a linear combination of the basis vectors. Dependence indicates the vectors do not span the entire space or are redundant.

Additional Resources

Consider The Bases B = {u_1, u_2} And B' = {u'_1, u'_2} For R^2, Where u_1 = [2, 2], u_2 = [4, -1], and u'_1 = [1, 3], u'_2 = [2, 1]. Understanding how different bases function in the two-dimensional real coordinate space (R^2) is fundamental in linear algebra, especially when it comes to changing coordinate systems, solving systems of equations, and understanding the structure of vector spaces. This article explores the concepts behind bases, their properties, and how to transition between different bases, providing insight into the practical applications and theoretical underpinnings of these mathematical constructs.

Foundations of Bases in R^2

What Is a Basis?

A basis of a vector space is a set of vectors that are both linearly independent and spanning. In simple terms, these vectors serve as building blocks for the entire space; any vector within R^2 can be expressed uniquely as a linear combination of the basis vectors.

Key properties:

- Linear independence: No vector in the basis can be written as a linear combination of the others.
- Spanning: Every vector in the space can be constructed using the basis vectors.

In R^2, any two vectors that are not scalar multiples of each other can form a basis. The two bases under consideration, B and B', are such sets.

The Specific Bases B and B'

Given:

- Basis B: $\{u_1, u_2\}$ where $u_1 = [2, 2]$, $u_2 = [4, -1]$
- Basis B': $\{u'_1, u'_2\}$ where $u'_1 = [1, 3]$, $u'_2 = [2, 1]$

Let's examine each basis more thoroughly.

Analyzing Basis B

Vectors:

- $u_1 = [2, 2]$
- $u_2 = [4, -1]$

Linear Independence:

To verify independence, check if one vector is a scalar multiple of the other:

- Is u_2 a scalar multiple of u_1?

Calculate the ratios:

- For the first component: 4 / 2 = 2
- For the second component: -1 / 2 = -0.5

Since these ratios are not equal, u_1 and u_2 are linearly independent, thus forming a valid basis.

Spanning R^2:

Any vector in R^2 can be expressed as a linear combination of u_1 and u_2. For example, the vector [x, y] can be written as:

$$ [x, y] = a \times [2, 2] + b \times [4, -1] $$

which leads to the system:

$$ 2a + 4b = x $$
$$ 2a - b = y $$

This system can be solved for a and b, confirming that u_1 and u_2 span R^2.

Analyzing Basis B'

Vectors:

- $u'_1 = [1, 3]$
- $u'_2 = [2, 1]$

Linear Independence:

Check if u'_2 is a multiple of u'_1:

- 2 / 1 = 2
- 1 / 3 ≈ 0.333

Since these ratios differ, u'_1 and u'_2 are linearly independent, forming another basis.

Spanning R^2:

Any vector [x, y] can be written as:

$$ [x, y] = c \times [1, 3] + d \times [2, 1] $$

leading to the system:

$$ c + 2d = x $$
$$ 3c + d = y $$

which again can be solved for c and d, confirming the basis property.

Change of Basis: From B to B'

Understanding how to convert coordinates from one basis to another is essential in linear algebra. This process is called a change of basis and involves constructing a transition matrix.

Constructing Transition Matrices

Transition from B to B'

To express vectors in B' in terms of B, or vice versa, we need the transition matrix.

Step 1: Write basis vectors as columns in matrices

- For B: \(M_B = [u_1 \quad u_2] = \begin{bmatrix} 2 & 4 \\ 2 & -1 \end{bmatrix} \)
- For B': \(M_{B'} = [u'_1 \quad u'_2] = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \)

Step 2: Find the transition matrix

The matrix that converts coordinates from B to B' is:

\[P_{B \rightarrow B'} = M_{B'}^{-1} \times M_B \]

Step 3: Compute \(M_{B'}^{-1} \)

Calculate the inverse of \(M_{B'} \):

\[
\det(M_{B'}) = (1)(1) - (3)(2) = 1 - 6 = -5
\]

\[
M_{B'}^{-1} = \frac{1}{\det} \times \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{-5} \times \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{bmatrix}
\]

Step 4: Calculate \(P_{B \rightarrow B'} \)

\[
P_{B \rightarrow B'} = M_{B'}^{-1} \times M_B
\]

\[
= \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{1}{5}

$$
\end{bmatrix} \times \begin{bmatrix} 2 & 4 \\ 2 & -1 \end{bmatrix}
$$

Calculating each entry:

- First row, first column:

$$
(-\frac{1}{5} \times 2) + (\frac{2}{5} \times 2) = -\frac{2}{5} + \frac{4}{5} = \frac{2}{5}
$$

- First row, second column:

$$
(-\frac{1}{5} \times 4) + (\frac{2}{5} \times -1) = -\frac{4}{5} - \frac{2}{5} = -\frac{6}{5}
$$

- Second row, first column:

$$
(\frac{3}{5} \times 2) + (-\frac{1}{5} \times 2) = \frac{6}{5} - \frac{2}{5} = \frac{4}{5}
$$

- Second row, second column:

$$
(\frac{3}{5} \times 4) + (-\frac{1}{5} \times -1) = \frac{12}{5} + \frac{1}{5} = \frac{13}{5}
$$

Thus,

$$
P_{B \rightarrow B'} = \begin{bmatrix} \frac{2}{5} & -\frac{6}{5} \\ \frac{4}{5} & \frac{13}{5} \end{bmatrix}
$$

This matrix allows us to convert the coordinates of any vector expressed in basis B into basis B'.

Practical Applications of Changing Bases

Coordinate Transformation in Computer Graphics

In computer graphics, objects are often manipulated in different coordinate systems. Converting between bases enables transformations such as rotations, scaling, and translations to be applied efficiently.

Solving Systems of Linear Equations

Changing bases can simplify the structure of a problem. For example, transforming to a basis where a matrix is diagonal makes solving systems more straightforward.

Data Compression and Signal Processing

Transforming data into different bases, such as Fourier or wavelet bases, can reveal features not visible in the original coordinate system, facilitating compression or noise reduction.

Visualizing the Bases

Visual intuition enhances understanding:

- Basis B: The vectors u_1 = [2, 2] and u_2 = [4, -1] form a parallelogram that covers R^2. Plotting these vectors shows they are not parallel and span the entire plane.

- Basis B': The vectors u'_1 = [1, 3] and u'_2 = [2, 1] also form a parallelogram, distinct from the first, but still covering R^2.

The Significance of Basis Choice

Choosing different bases impacts how we interpret and manipulate vectors:

- Coordinate systems: Different bases provide alternative perspectives.
- Simplification: Some bases diagonalize matrices, simplifying computations.
- Flexibility: Transitioning between bases allows for versatile problem-solving approaches.

Understanding the transition matrices and the properties of bases equips mathematicians,

Consider The Bases B U1 U2 And B U 1 U 2 For R2 Where U1 2 2 U2 4 1 U1 1

Consider The Bases B U1 U2 And B U 1 U 2 For R2 Where U1 2 2 U2 4 1 U1 1

Related Articles

- Bramble Corp. Reported Net Income Of $522000 For The Year Ended 12/31/21. Included In The Computation
- Cruises Are Expensive. You Can Take Steps That Will Help You Save Money And Get The Most Out Of Every
- Elena Went To A Store Where You Can Scoop Your Own Popcorn And Buy As Much As You Want. She Bought 10

Back to Home