Consider The Following. (2 + X^2)y'' - Xy' + 4y = 0, X_0 = 0 Seek Power Series Solutions Of The Given

Consider The Following. (2 + X^2)y'' - Xy' + 4y = 0, X_0 = 0 Seek Power Series Solutions Of The Given

Understanding and solving differential equations is a fundamental aspect of advanced mathematics, physics, and engineering. Among various methods, the power series approach is particularly powerful for solving linear differential equations with variable coefficients. In this article, we explore the process of finding power series solutions to the differential equation:

$(2 + X^2) y'' - X y' + 4 y = 0$, with the initial point $X_0 = 0$

This equation presents an interesting case due to its variable coefficients, making direct methods less straightforward. We will systematically develop the power series solution, analyze the recurrence relations, and discuss the convergence and implications of the solutions obtained.

Understanding the Differential Equation

Equation Overview

The given differential equation is:

$(2 + X^2) y'' - X y' + 4 y = 0$

This is a second-order linear ordinary differential equation (ODE) with variable coefficients, where the coefficients depend explicitly on X. The initial point $X_0 = 0$ suggests that we are interested in solutions centered around X = 0, making it suitable for a power series expansion about this point.

Significance of Variable Coefficients

Variable coefficients often prevent straightforward solutions using elementary functions. Instead, power series methods allow us to construct solutions locally around a point of interest, providing insights into the behavior of solutions near that point.

Methodology for Power Series Solutions

Assuming a Power Series Solution

We seek a solution in the form of a power series centered at $X=0$:

$$y(X) = \sum_{n=0}^{\infty} a_n X^n$$

where a_n are coefficients to be determined.

Similarly, derivatives are expressed as:

$$y'(X) = \sum_{n=1}^{\infty} n a_n X^{n-1}$$

$$y''(X) = \sum_{n=2}^{\infty} n(n-1) a_n X^{n-2}$$

Substituting into the Differential Equation

Substituting the power series into the differential equation involves expressing $(2 + X^2) y''$, $X y'$, and $4 y$, all as power series:

1. $(2 + X^2) y'' = 2 y'' + X^2 y''$
2. $-X y'$ remains as is
3. $4 y$ remains as is

Each term is expanded as a power series, and then all are combined to match coefficients of like powers of X.

Deriving the Recurrence Relation

Expressing Each Term

Let's write each term explicitly:

- Term 1: $2 y'' = 2 \sum_{n=2}^{\infty} n(n-1) a_n X^{n-2} = \sum_{n=2}^{\infty} 2 n(n-1) a_n X^{n-2}$

- Term 2: $X^2 y'' = X^2 \sum_{n=2}^{\infty} n(n-1) a_n X^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n$

X^{n}

- Term 3: $-X y' = -X \Sigma_{n=1}^\infty n a_n X^{n-1} = - \Sigma_{n=1}^\infty n a_n X^{n}$

- Term 4: $4 y = 4 \Sigma_{n=0}^\infty a_n X^n$

Reindexing Series for Consistency

To combine these series, we need all sums expressed with the same powers of X:

- For the first term, set $m = n - 2$:

When $n=2$, $m=0$; as $n\to\infty$, $m\to\infty$

So, $2 y'' = \Sigma_{m=0}^\infty 2 (m+2)(m+1) a_{m+2} X^{m}$

- For the second term, n remains as is:

$\Sigma_{n=2}^\infty n(n-1) a_n X^{n} = \Sigma_{n=2}^\infty n(n-1) a_n X^{n}$

- For the third term, n remains as is:

- $\Sigma_{n=1}^\infty n a_n X^{n}$

- For the fourth term:

$4 \Sigma_{n=0}^\infty a_n X^{n}$

Now, the entire differential equation becomes:

$\Sigma_{m=0}^\infty 2 (m+2)(m+1) a_{m+2} X^{m} + \Sigma_{n=2}^\infty n(n-1) a_n X^{n} - \Sigma_{n=1}^\infty n a_n X^{n} + 4 \Sigma_{n=0}^\infty a_n X^{n} = 0$

Aligning the Series

Express all sums over X^{n} or X^{m} with the same index:

- For the second sum, start from $n=0$:

When $n=2$, sum begins; for $n=0$, term is zero because $n(n-1)=0$

- For the third sum, $n=1$ to ∞

- For the last sum, $n=0$ to ∞

Expressed uniformly:

- First sum: $m=0$ to ∞

- Second sum: $n=2$ to ∞

- Third sum: $n=1$ to ∞

- Fourth sum: $n=0$ to ∞

Formulating the Recurrence Relations

Equating Coefficients

Since the sum equals zero for all X, each coefficient of X^n must be zero:

For each power $n \geq 0$,

Coefficient sum:

$[2 (n+2)(n+1) a_{n+2}] + [n(n-1) a_n \text{ if } n \geq 2] - [n a_n \text{ if } n \geq 1] + [4 a_n] = 0$

Note that for $n=0$:

- The second sum does not contribute (starts from $n=2$)

- The third sum: $n=1$, so for $n=0$, no contribution

- The fourth sum: $n=0$

Thus, for $n=0$:

$2 (2)(1) a_2 + 4 a_0 = 0$

which simplifies to:

$2\ 2\ 1\ a_2 + 4 a_0 = 0$

$4 a_2 + 4 a_0 = 0$

$\Rightarrow a_2 = - a_0$

Similarly, for $n \geq 1$, the general recurrence relation is:

$2 (n+2)(n+1) a_{n+2} + [n(n-1) - n + 4] a_n = 0$

which simplifies to:

$$2 (n+2)(n+1) a_{n+2} + (n^2 - n + 4) a_n = 0$$

Final Recurrence Formula

Thus, the recurrence relation for $n \geq 0$ is:

$$a_{n+2} = - \left[(n^2 - n + 4) / (2 (n+2)(n+1)) \right] a_n$$

with initial coefficients a_0 and a_1 remaining arbitrary (as per the nature of second-order linear ODEs).

Constructing the Power Series Solution

Explicit Expressions for Coefficients

Using the recurrence:

$$a_{n+2} = - \left[(n^2 - n + 4) / (2 (n+2)(n+1)) \right] a_n$$

and initial values a_0, a_1, we can compute subsequent coefficients iteratively:

- $a_2 = - a_0$ (from earlier)

- $a_3 = - [(1^2 - 1 + 4) / (2\ 3\ 2)] a_1 = - [(1 -1 +4)/(12)] a_1 = - (4/12) a_1 = - (1/3) a_1$

- $a_4 = - [(2^2 - 2 + 4) / (2\ 4\ 3)] a_2 = - [(4 - 2 + 4)/24] a_2 = - (6/24) a_2 = - (1/4) a_2$

But since $a_2 = - a_0$, we substitute back:

$$a_4 = - (1/4)(- a_0) = (1/4) a_0$$

Similarly, higher coefficients are obtained recursively.

Series Solution Structure

The general solution is a linear combination:

$$y(X) = a_0 \Sigma_{k=0}^\infty c_{2k} X^{2k} + a_1 \Sigma_{k=0}^\infty c_{2k+1} X^{2k+1}$$

where the coefficients c_{n} are computed via the recurrence relation, and a_0, a_1 are arbitrary constants.

Analysis of

Frequently Asked Questions

What is the main goal when seeking power series solutions for the differential equation $(2 + x^2)y'' - xy' + 4y = 0$ with the initial condition $x_0 = 0$?

The primary goal is to find a solution y(x) expressed as a power series expansion around x = 0 that satisfies the differential equation and initial condition, enabling us to analyze the behavior of solutions near the point x = 0.

How do you set up the power series solution for the differential equation involving $(2 + x^2)y'' - xy' + 4y = 0$?

You assume a solution of the form $y(x) = \Sigma a_n x^n$, where the sum runs from n=0 to infinity. Then, compute y', y'', substitute into the differential equation, and equate coefficients of like powers of x to find recurrence relations for the coefficients a_n.

What is the significance of the initial condition x_0 = 0 in solving this differential equation using power series?

The initial condition x_0 = 0 indicates the expansion is centered at x = 0, which simplifies the series expansion and allows us to determine the coefficients a_n based on initial values y(0) and y'(0).

How do the coefficients of the power series relate to the recurrence relations derived from the differential equation?

The recurrence relations express each coefficient a_n in terms of previous coefficients, allowing iterative computation of all coefficients once initial values are known, ultimately leading to the full power series solution.

Are there any special functions or known solutions that arise from solving this differential equation via power series?

Depending on the recurrence relations obtained, the solutions may relate to special functions such as hypergeometric functions or Bessel functions, especially if the series can be

summed or recognized in standard forms.

What challenges might one face when solving $(2 + x^2)y'' - xy' + 4y = 0$ using power series methods?

Challenges include handling the variable coefficient $(2 + x^2)$, which complicates the recurrence relations, and ensuring convergence of the power series near the expansion point. Additionally, identifying closed-form solutions might be non-trivial.

How can the initial condition $x_0 = 0$ influence the convergence and validity of the power series solution?

Since the series is centered at $x = 0$, the initial condition helps determine the series coefficients directly. Moreover, it ensures the radius of convergence includes $x = 0$, making the solution valid in a neighborhood around the initial point.

Additional Resources

Consider The Following. $(2 + X^2)y'' - Xy' + 4y$

= 0, X_0 = 0 Seek Power Series Solutions Of The Given

Introduction

In the realm of differential equations, finding solutions that are both manageable and insightful often involves employing series methods, especially when dealing with equations that resist straightforward integration. The differential equation

$$(2 + X^2)\, y'' - X\, y' + 4\, y = 0$$

with the initial point $X_0 = 0$, exemplifies such a challenge. Its structure suggests a non-constant coefficient differential equation that is not readily solvable through elementary methods. Instead, the power series approach provides a powerful tool to unravel its solutions near the point of interest, X = 0.

This article aims to thoroughly analyze this equation, exploring the process of seeking its solutions in the form of power series expansions. We will examine the theoretical

underpinnings, derive the recurrence relations, and interpret the nature of the solutions obtained. Ultimately, this enriches our understanding of the behavior of solutions around X=0 and demonstrates the depth and applicability of power series techniques in differential equations.

Background and Motivation

Before delving into the solution process, it is essential to understand why power series methods are appropriate here. The given differential equation is a second-order linear ODE with variable coefficients, which often precludes simple solutions in terms of elementary functions. Near a regular point (here, X=0), power series expansions can provide local solutions, offering insights into the behavior of the system.

Furthermore, such equations often appear in physics, engineering, and applied mathematics, where solutions near specific points are crucial, such as in quantum mechanics, wave propagation, or stability analysis. The

method's flexibility allows for approximate solutions that can be refined as needed.

Formulation of the Power Series Solution

Assuming a Power Series Expansion

To seek power series solutions near X=0, we posit that the solution y(X) can be expressed as:

$$y(X) = \sum_{n=0}^{\infty} a_n X^{n+r}$$

where:

- a_n are coefficients to be determined,
- r is the indicial exponent, which accounts for potential singular behavior at X=0.

The choice of the form involving r (the Frobenius method) is particularly relevant if

the point X=0 is a regular singular point. To verify this, we analyze the equation's coefficients near X=0.

Determining the Nature of the Point X=0

The standard form of a second-order linear ODE is:

$$y'' + P(X)\, y' + Q(X)\, y = 0$$

which can be derived by dividing the entire equation by $(2 + X^2)$:

$$y'' - [X / (2 + X^2)]\, y' + [4 / (2 + X^2)]\, y = 0$$

At X=0:

- $P(X) = -X / (2 + X^2) \rightarrow P(0)=0$,
- $Q(X) = 4 / (2 + X^2) \rightarrow Q(0)=2$.

Since both P(X) and Q(X) are analytic at X=0 (i.e., expandable in power series around zero),

the point $X=0$ is an ordinary point, not a singular point. Therefore, the Frobenius method simplifies to assuming a power series without an indicial exponent r, i.e., $r=0$, simplifying the solution to:

$$y(X) = \sum_{n=0}^\infty a_n X^n$$

Derivation of the Power Series Solution

Calculating Derivatives

Assuming:

$$y(X) = \sum_{n=0}^\infty a_n X^n$$

then:

$$y'(X) = \sum_{n=1}^\infty n a_n X^{n-1}$$

and

$y''(X) = \sum_{n=2}$^∞ $n(n-1)\ a_n\ X^{n-2}$

Substituting into the Differential Equation

Our original equation:

$(2 + X^2)\ y'' - X\ y' + 4\ y = 0$

becomes:

$(2 + X^2) \sum_{n=2}$^∞ $n(n-1)\ a_n\ X^{n-2} - X \sum_{n=1}$^∞ $n\ a_n\ X^{n-1}$ +
$4 \sum_{n=0}$^∞ $a_n\ X^n = 0$

Distribute:

$2 \sum_{n=2}$^∞ $n(n-1)\ a_n\ X^{n-2} + X^2 \sum_{n=2}$^∞ $n(n-1)\ a_n\ X^{n-2}$ -

$\sum_{n=1}$^∞ n a_n X^n + 4 $\sum_{n=0}$^∞ a_n X^n = 0

Rewrite each term to align powers of X:

- First term:

2 $\sum_{n=2}$^∞ n(n-1) a_n X^{n-2}
= 2 $\sum_{k=0}$^∞ (k+2)(k+1) a_{k+2} X^k

by substituting n=k+2.

- Second term:

X^2 $\sum_{n=2}$^∞ n(n-1) a_n X^{n-2} = $\sum_{n=2}$^∞ n(n-1) a_n X^n
= $\sum_{k=0}$^∞ (k+2)(k+1) a_{k+2} X^{k+2}

But wait! Since the sum originally is over n, with the substitution n=k+2, then the term becomes:

$\sum_{k=0}$^∞ (k+2)(k+1) a_{k+2} X^{k+2}

which is better expressed as:

$$\sum_{k=0}^\infty (k+2)(k+1)\, a_{k+2}\, X^{k+2}$$

But note that the power is k+2, so to align with previous sums, we factor out X^2:

Similarly, the third term:

$$- \sum_{n=1}^\infty n\, a_n\, X^n$$

and the fourth term:

$$4 \sum_{n=0}^\infty a_n\, X^n$$

Now, rewriting all sums in powers of X:

\[
\begin{aligned}
& 2 \sum_{k=0}^\infty (k+2)(k+1) a_{k+2} X^{k}
\\
& + \sum_{k=0}^\infty (k+2)(k+1) a_{k+2}

X^{k+2} \\
& - \sum_{n=1}^\infty n a_{n} X^{n} \\
& + 4 \sum_{n=0}^\infty a_{n} X^{n} = 0
\end{aligned}
\]

We want to write all sums with the same power of X, say X^k. The second sum involves X<

Consider The Following 2 X 2 Y Xy 4y 0 X 0 0 Seek Power Series Solutions Of The Given

Consider The Following 2 X 2 Y Xy 4y 0 X 0 0 Seek Power Series Solutions Of The Given

Related Articles

- Consider The Length-tension Relationship. Then, Click And Drag Each Statement Into The Correct Box To
- Blocks A And B Of Masses M And 2m, Respectively, Are Connected By A Light String And Are Pulled Along
- Determine The Apparent Weight (as A Multiple Of Their Weight Mg) For A Rider In The Fourth Car At The

Back to Home