

Consider The Following Differential Equation: $\frac{dy}{dx} + \tan(x)y = \sec(x)$. The Equation Can Be Solved By

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Differential equations are fundamental in mathematical modeling across various scientific and engineering disciplines. Among these, first-order linear differential equations hold particular significance due to their widespread occurrence and the availability of systematic solution methods. One such equation is:

$$\frac{dy}{dx} + \tan(x)y = \sec(x)$$

This differential equation presents a classic linear form that can be tackled effectively using integrating factors. In this comprehensive guide, we will explore the process of solving this differential equation step-by-step, elucidate the underlying concepts, and provide insights into related solution techniques. Whether you're a student delving into differential equations for the first time or a professional brushing up on solving methods, this article offers valuable information.

Understanding the Structure of the Differential Equation

Before diving into the solution process, it's essential to recognize the form of the differential equation. The general first-order linear differential equation can be written as:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

In our case:

- $P(x) = \tan(x)$
- $Q(x) = \sec(x)$

This form allows us to apply the method of integrating factors, a powerful technique for solving linear equations of this type.

Method of Solving: Integrating Factor Technique

The primary approach for solving the differential equation $\frac{dy}{dx} + P(x)y = Q(x)$ involves multiplying through by an integrating factor $\mu(x)$, which simplifies the left side into an exact derivative.

Step 1: Find the Integrating Factor $\mu(x)$

The integrating factor $\mu(x)$ is given by:

$$\mu(x) = e^{\int P(x) dx}$$

For our equation:

$$\mu(x) = e^{\int \tan(x) \, dx}$$

Recall that:

$$\int \tan(x) \, dx = -\ln |\cos(x)| + C$$

Therefore,

$$\mu(x) = e^{-\ln |\cos(x)|} = \frac{1}{|\cos(x)|}$$

Since the absolute value is not critical in indefinite integrals, and assuming $(\cos(x) > 0)$ in the interval of interest, we can write:

$$\mu(x) = \frac{1}{\cos(x)} = \sec(x)$$

Step 2: Multiply the Entire Equation by $(\mu(x))$

Multiplying both sides by $(\sec(x))$:

$$\sec(x) \frac{dy}{dx} + \sec(x) \tan(x) y = \sec^2(x)$$

Observe that:

$$\frac{d}{dx} (\sec(x) y) = \sec(x) \frac{dy}{dx} + \sec(x) \tan(x) y = \frac{d}{dx} (\sec(x) y)$$

This is because:

$$\frac{d}{dx} (\sec(x) y) = \sec(x) \frac{dy}{dx} + y \frac{d}{dx} (\sec(x))$$

and since:

$$\frac{d}{dx} \sec(x) = \sec(x) \tan(x)$$

the product rule confirms the equivalence.

Step 3: Rewrite the Equation as an Exact Derivative

The differential equation now simplifies to:

$$\frac{d}{dx} (\sec(x) y) = \sec^2(x)$$

This is a significant simplification because integrating the right side becomes straightforward.

Integrating Both Sides

To find $y(x)$, integrate both sides with respect to x :

$$\int \frac{d}{dx} (\sec(x) y) dx = \int \sec^2(x) dx$$

which yields:

$$\sec(x) y = \int \sec^2(x) dx + C$$

Recall that:

$$\int \sec^2(x) dx = \tan(x) + C$$

Thus, the general solution is:

$$\sec(x) y = \tan(x) + C$$

where C is an arbitrary constant.

Solving for y : Final Expression

To explicitly express y :

$$y = \frac{\tan(x) + C}{\sec(x)} = (\tan(x) + C) \cos(x)$$

Using the trigonometric identity:

$$\boxed{\text{General solution: } y(x) = \tan(x) \cos(x) + C \cos(x)}$$

Note that $\tan(x) \cos(x) = \sin(x)$, hence:

$$\boxed{y(x) = \sin(x) + C \cos(x)}$$

This simplified form offers a clear understanding of the solution's behavior in relation to the original differential equation.

Summary of Solution Methodology

To summarize, solving the differential equation $\frac{dy}{dx} + \tan(x) y = \sec(x)$ involves the following key steps:

- 1. Identify the form:** Recognize it as a linear first-order differential equation with $P(x) = \tan(x)$ and $Q(x) = \sec(x)$.
- 2. Calculate the integrating factor:** $\mu(x) = e^{\int P(x) dx} = \sec(x)$.
- 3. Multiply through by the integrating factor:** Simplify the equation to an exact derivative: $\frac{d}{dx} (\sec(x) y) = \sec^2(x)$.
- 4. Integrate both sides:** Obtain $\sec(x) y = \tan(x) + C$.
- 5. Solve for y :** $y = (\tan(x) + C)/\sec(x) = \sin(x) + C \cos(x)$.

Additional Insights and Applications

Understanding how to solve this type of differential equation is crucial for various applications, including:

- Physics:** Modeling oscillations and wave phenomena where parameters depend on trigonometric functions.

- **Engineering:** Signal processing involving sinusoidal signals and their differential relationships.
- **Mathematics:** Developing intuition for integrating factors and applying them to more complex equations.

Moreover, the approach demonstrated here can be generalized to other linear differential equations with different $P(x)$ and $Q(x)$, emphasizing the versatility of the integrating factor method.

Common Mistakes to Avoid

While solving differential equations using integrating factors is straightforward, several pitfalls can hinder correct solutions:

1. **Incorrect Integration of $P(x)$:** Ensure proper integration of $P(x)$. For $\tan(x)$, recall the integral involves a natural logarithm.
2. **Ignoring Absolute Values:** When integrating, be cautious with absolute values in logarithms, especially over different intervals.
3. **Overlooking the Domain:** The solution involves $\sec(x)$ and $\tan(x)$, which are undefined at certain points; always specify the domain of the solution.

Conclusion

Solving the differential equation $\frac{dy}{dx} + \tan(x)y = \sec(x)$ exemplifies the effectiveness of the integrating factor method for linear first-order differential equations. By recognizing the structure, calculating the integrating factor, and performing straightforward integrations, we derive the general solution:

$$\boxed{y(x) = \sin(x) + C \cos(x)}$$

This process not only underscores the importance of systematic approaches but also highlights the deep interconnection between differential equations and trigonometric identities. Mastery of such techniques enhances problem-solving skills and equips practitioners to handle a wide array of real-world problems modeled by differential equations.

Frequently Asked Questions

What is the form of the differential equation $Dy/dx + \tan(x)y = \sec(x)$?

It is a first-order linear differential equation in the form $Dy/dx + P(x)y = Q(x)$, where $P(x) = \tan(x)$ and $Q(x) = \sec(x)$.

How can we identify the method to solve the differential equation

$$Dy/dx + \tan(x)y = \sec(x)?$$

Since it is a linear differential equation, it can be solved using an integrating factor method.

$$\text{What is the integrating factor for the differential equation } Dy/dx + \tan(x)y = \sec(x)?$$

The integrating factor is $\mu(x) = e^{\int \tan(x) dx} = e^{\{-\ln|\cos(x)|\}} = 1/|\cos(x)|$.

$$\text{How do we compute the integrating factor for this differential equation?}$$

By calculating $\mu(x) = e^{\int \tan(x) dx}$ which simplifies to $1/|\cos(x)|$, since $\int \tan(x) dx = -\ln|\cos(x)|$.

$$\text{What is the next step after finding the integrating factor for this differential equation?}$$

Multiply the entire differential equation by the integrating factor to rewrite it as a perfect derivative, allowing integration on both sides.

$$\text{How do you solve the differential equation } Dy/dx + \tan(x)y = \sec(x) \text{ using the integrating factor?}$$

Multiply both sides by $1/|\cos(x)|$, then recognize the left side as the derivative of $y/\cos(x)$, and integrate both sides with respect to x .

$$\text{What is the general solution to the differential equation } Dy/dx + \tan(x)y = \sec(x)?$$

The general solution is $y(x) = \sin(x) (\int \sec^2(x) dx + C)$, which simplifies to $y(x) = \sin(x) (\tan(x) + C)$.

How does recognizing the integrating factor simplify solving the differential equation?

It allows rewriting the left side as a derivative of a product, enabling straightforward integration and solution derivation.

Can the differential equation $Dy/dx + \tan(x)y = \sec(x)$ be classified as a separable differential equation?

No, it is a linear first-order differential equation, not separable, but solvable via the integrating factor method.

What are common applications of solving differential equations like $Dy/dx + \tan(x)y = \sec(x)$?

Such equations appear in physics, engineering, and mathematics, especially in modeling systems with variable coefficients or growth processes.

Additional Resources

Consider The Following Differential Equation: $Dy/dx + \tan(x)y = \sec(x)$. The Equation Can Be Solved By

Differential equations are fundamental tools in mathematics, physics, engineering, and numerous scientific disciplines. They describe how a quantity changes relative to another, often encapsulating complex systems in manageable forms. Among these, first-order linear differential equations are particularly prevalent, offering pathways to solutions that reveal the behavior of systems across various contexts.

Today, we explore a specific first-order linear differential equation:

$$\left[\frac{dy}{dx} + \tan(x) y = \sec(x) \right]$$

This equation may seem intimidating at first glance, but beneath its surface lies a systematic method for solving it—namely, the integrating factor technique. By carefully dissecting each component and applying established methods, we can uncover the general solution and appreciate the elegance of differential equation solving.

Understanding the Structure of the Differential Equation

Before diving into solution methods, it's essential to analyze the structure of the given equation:

$$\left[\frac{dy}{dx} + P(x) y = Q(x) \right]$$

In this case:

$$- \left(P(x) = \tan(x) \right)$$

$$- \left(Q(x) = \sec(x) \right)$$

This is a linear first-order differential equation, which generally can be solved using the integrating factor method. Recognizing this form allows us to leverage a well-established strategy to find solutions.

The Integrating Factor Method: A Step-by-Step Guide

The integrating factor method transforms the original equation into an exact differential, enabling straightforward integration. Let's outline the steps:

Step 1: Write the Standard Form

Ensure the differential equation is in the form:

$$\left[\frac{dy}{dx} + P(x)y = Q(x) \right]$$

Our equation already fits this form:

$$\left[\frac{dy}{dx} + \tan(x)y = \sec(x) \right]$$

Step 2: Calculate the Integrating Factor $\mu(x)$

The integrating factor is given by:

$$\left[\mu(x) = e^{\int P(x) dx} \right]$$

Calculate:

$$\left[\int \tan(x) dx \right]$$

Recall the integral:

$$\left[\int \tan(x) dx = -\ln|\cos(x)| + C \right]$$

Ignoring the constant (since it cancels out in the exponent), we have:

$$\left[\mu(x) = e^{-\ln|\cos(x)|} = \frac{1}{|\cos(x)|} \right]$$

For simplicity, and because $|\cos(x)|$ appears in the denominator, we can write:

$$\left[\mu(x) = \frac{1}{\cos(x)} = \sec(x) \right]$$

(Note: For the purpose of solving, we assume x is in an interval where $\cos(x) \neq 0$.)

Step 3: Multiply the Entire Equation by the Integrating Factor

Multiply through by $\mu(x) = \sec(x)$:

$$\left[\sec(x) \frac{dy}{dx} + \sec(x) \tan(x) y = \sec^2(x) \right]$$

Observe that:

$$\left[\sec(x) \tan(x) = \frac{\sin(x)}{\cos^2(x)} \right]$$

which simplifies the expression, but more importantly, note that the left side becomes the derivative of $(y \cdot \mu(x))$:

$$\left[\frac{d}{dx} [y \cdot \sec(x)] = \sec^2(x) \right]$$

This step leverages the product rule:

$$\left[\frac{d}{dx} [y \cdot \sec(x)] = y' \sec(x) + y \frac{d}{dx} \sec(x) \right]$$

Since:

$$\left[\frac{d}{dx} \sec(x) = \sec(x) \tan(x) \right]$$

the derivative matches our multiplied equation exactly.

Solving the Transformed Equation

Now, integrate both sides with respect to x :

$$\int \frac{d}{dx} [y \cdot \sec(x)] = \sec^2(x) \, dx$$

Integrate:

$$y \cdot \sec(x) = \int \sec^2(x) \, dx + C$$

Recall the integral:

$$\int \sec^2(x) \, dx = \tan(x) + C$$

Therefore:

$$y \cdot \sec(x) = \tan(x) + C$$

Finally, solve for y :

$$y = \cos(x) [\tan(x) + C]$$

or, equivalently,

$$y = \sin(x) + C \cos(x)$$

The General Solution and Its Implications

The general solution to the differential equation is:

$$y = \sin(x) + C \cos(x)$$

$$\boxed{y(x) = \sin(x) + C \cos(x)}$$

where C is an arbitrary constant determined by initial conditions.

This solution encapsulates the behavior of y across the domain where the functions are defined (excluding points where $\cos(x) = 0$), i.e., $x = \frac{\pi}{2} + n\pi$. The structure highlights the interplay between trigonometric functions and differential equations, serving as a reminder of the interconnectedness of mathematical concepts.

Practical Applications and Significance

Understanding how to solve such equations is more than an academic exercise. Differential equations like this appear in:

- Electrical Engineering: Modeling circuits with sinusoidal signals.
- Physics: Describing oscillatory systems such as pendulums or waves.
- Biology: Modeling periodic phenomena like circadian rhythms.
- Economics: Analyzing cyclic trends over time.

Mastering the integrating factor method equips scientists and engineers with a versatile tool for tackling a wide array of real-world problems involving change and dynamics.

Additional Tips for Solving Similar Equations

- Identify the form: Ensure the differential equation is linear and first-order.
- Compute the integrating factor carefully: Pay attention to integral calculations, especially with trigonometric functions.
- Check domain restrictions: Be aware of points where functions like $\cos(x)$ or $\sin(x)$ are zero, affecting the domain of the solution.
- Verify solutions: Substitute back into the original equation to confirm correctness.
- Practice with variations: Try equations with different $P(x)$ and $Q(x)$ functions to deepen understanding.

Conclusion

The differential equation $\frac{dy}{dx} + \tan(x) y = \sec(x)$ exemplifies the elegance and utility of the integrating factor method. By systematically applying the steps—identifying the form, computing the integrating factor, transforming the equation, and integrating—we arrive at a solution that not only solves the equation but also reveals the underlying relationships between trigonometric functions and differential calculus.

Such methods form the backbone of mathematical modeling in sciences and engineering, enabling practitioners to interpret, predict, and manipulate systems governed by change. As you deepen your understanding and practice solving various differential equations, you'll uncover more of the beautiful structure underlying the language of change that mathematics provides.

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