Consider The Following IVP: U"(t) + U'(t) - 12u(t)=0 (1) U(0) = 70 And U'(0) = 18. Show That U(t)=ce"

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Introduction to the Initial Value Problem (IVP)

In the realm of differential equations, initial value problems (IVPs) serve as foundational tools for modeling real-world phenomena across physics, engineering, and other scientific disciplines. The specific IVP under consideration involves a second-order linear differential equation:

$$ U''(t) + U'(t) - 12 U(t) = 0 $$

along with the initial conditions:

$$ U(0) = 70 \quad \text{and} \quad U'(0) = 18 $$

The goal is to demonstrate that the solution $U(t)$ can be expressed in the form:

$$ U(t) = c e^{\lambda t} $$

where c and λ are constants to be determined. This process involves solving the differential equation systematically and applying the initial conditions to find the specific solution.

Understanding the Differential Equation

Type of Differential Equation

The given differential equation:

$$ U''(t) + U'(t) - 12 U(t) = 0 $$

is a homogeneous linear second-order differential equation with constant coefficients. Such equations are prevalent in modeling oscillations, exponential growth/decay, and other dynamic systems.

Significance of the Equation

- Homogeneity: The right-hand side is zero, indicating no external forcing.
- Linearity: The superposition principle applies, meaning solutions can be added.
- Constant coefficients: Simplifies the solution process via characteristic equations.

Solving the Differential Equation

Step 1: Write the Characteristic Equation

To solve the differential equation, assume a solution of the form:

$$U(t) = e^{\lambda t}$$

Substituting into the differential equation yields:

$$\lambda^2 e^{\lambda t} + \lambda e^{\lambda t} - 12 e^{\lambda t} = 0$$

Dividing through by $e^{\lambda t}$ (which is never zero):

$$\lambda^2 + \lambda - 12 = 0$$

This quadratic equation, known as the characteristic equation, determines the form of the general solution.

Step 2: Find the Roots of the Characteristic Equation

Solve:

$$\lambda^2 + \lambda - 12 = 0$$

Using the quadratic formula:

$$ \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} $$

where $a=1$, $b=1$, and $c=-12$:

$$ \lambda = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times (-12)}}{2} $$
$$ \lambda = \frac{-1 \pm \sqrt{1 + 48}}{2} $$
$$ \lambda = \frac{-1 \pm \sqrt{49}}{2} $$
$$ \lambda = \frac{-1 \pm 7}{2} $$

Thus, the roots are:

- $\lambda_1 = \frac{-1 + 7}{2} = 3$
- $\lambda_2 = \frac{-1 - 7}{2} = -4$

Step 3: Write the General Solution

Since the roots are real and distinct, the general solution is:

$$ U(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} $$
$$ U(t) = C_1 e^{3t} + C_2 e^{-4t} $$

where C_1 and C_2 are arbitrary constants determined by initial conditions.

Applying Initial Conditions to Find Specific Solution

Step 4: Compute $U'(t)$

Differentiate $U(t)$:

$$ U'(t) = 3 C_1 e^{3t} - 4 C_2 e^{-4t} $$

Step 5: Use Initial Conditions

Given:

$$ U(0) = 70 $$
$$ U'(0) = 18 $$

Plugging in $t=0$:

$$ U(0) = C_1 e^{0} + C_2 e^{0} = C_1 + C_2 $$
$$ 70 = C_1 + C_2 \quad \Rightarrow \quad (1) $$

Similarly, for $U'(0)$:

$$ U'(0) = 3 C_1 e^{0} - 4 C_2 e^{0} = 3 C_1 - 4 C_2 $$
$$ 18 = 3 C_1 - 4 C_2 \quad \Rightarrow \quad (2) $$

Step 6: Solve for C_1 and C_2

From equation (1):

$$ C_2 = 70 - C_1 $$

Substitute into equation (2):

$$ 18 = 3 C_1 - 4(70 - C_1) $$
$$ 18 = 3 C_1 - 280 + 4 C_1 $$
$$ 18 = 7 C_1 - 280 $$

Add 280 to both sides:

$$ 298 = 7 C_1 $$
$$ C_1 = \frac{298}{7} \approx 42.57 $$

Now, find C_2:

$$ C_2 = 70 - \frac{298}{7} = \frac{490}{7} - \frac{298}{7} = \frac{192}{7} \approx 27.43 $$

Final Solution and Expression

The specific solution to the initial value problem is:

$$ U(t) = \frac{298}{7} e^{3t} + \frac{192}{7} e^{-4t} $$

which can be written as:

$$ U(t) = C_1 e^{3t} + C_2 e^{-4t} $$

with explicit constants:

- $C_1 = \frac{298}{7}$
- $C_2 = \frac{192}{7}$

Showing the Solution in the Form $U(t) = c e^{\lambda t}$

While the general solution involves two exponential terms, the statement "Show that $U(t) = c e^{\lambda t}$" suggests examining particular solutions or specific cases where the solution reduces to a single exponential form.

Case 1: When the roots are equal

This occurs if the characteristic equation has a repeated root:

$$ \lambda^2 + \lambda - 12 = 0 $$

has discriminant:

$$ \Delta = 1^2 - 4 \times 1 \times (-12) = 1 + 48 = 49 \neq 0 $$

Hence, roots are distinct, and the solution is a linear combination of two exponentials, not a single exponential.

Case 2: When the solution simplifies to a single exponential

Suppose we consider specific initial conditions that lead to one of the constants being zero:

- If $C_2 = 0$, then $U(t) = C_1 e^{3t}$
- If $C_1 = 0$, then $U(t) = C_2 e^{-4t}$

In the context of the initial conditions, this is not the case, as both constants are non-zero.

Conclusion

Therefore, the general solution involves a linear combination of two exponential functions, and the specific solution determined by initial conditions does not reduce to a single exponential of the form $c e^{\lambda t}$. However, in theoretical contexts, particular solutions of the form $U(t) = c e^{\lambda t}$ are fundamental, especially when considering homogeneous equations with a single characteristic root or when analyzing particular modes.

Summary and Final Remarks

- The differential equation $U''(t) + U'(t) - 12 U(t) = 0$ is a second-order linear homogeneous differential equation with constant coefficients.
- The characteristic equation yields roots $\lambda = 3$ and $\lambda = -4$.
- The general solution is a linear combination:

$$U(t) = C_1 e^{3t} + C_2 e^{-4t}$$

- Applying initial conditions $U(0) = 70$ and $U'(0) = 18$, we find the specific constants:

$$C_1 = \frac{298}{7}, \quad C_2 = \frac{192}{7}$$

- The solution can be expressed explicitly as:

\[U(t) = \frac{298}{7} e^{3t} +

Frequently Asked Questions

What is the characteristic equation associated with the differential equation U''(t) + U'(t) - 12U(t) = 0?

The characteristic equation is r^2 + r - 12 = 0.

How do we find the general solution to the differential equation $U''(t) + U'(t) - 12U(t) = 0$?

Solve the characteristic equation for r, then write the general solution as $U(t) = C1\, e^{r1\, t} + C2\, e^{r2\, t}$, where r1 and r2 are the roots.

What are the roots of the characteristic equation $r^2 + r - 12 = 0$?

The roots are $r = 3$ and $r = -4$, obtained by factoring or using the quadratic formula.

How do we determine the constants C1 and C2 using the initial conditions $U(0) = 70$ and $U'(0) = 18$?

Substitute t=0 into the general solution and its derivative to create a system of equations, then solve for C1 and C2.

What is the specific solution for U(t) given the initial conditions, and how does it relate to the form $c\, e^{rt}$?

The solution is $U(t) = C1\, e^{3t} + C2\, e^{-4t}$, with constants C1 and C2 determined from initial conditions; the statement $U(t) = c\, e^{...}$ suggests a particular exponential form, possibly for a specific solution or part of the general solution.

Additional Resources

Comprehensive Analysis of the Initial Value Problem $U''(t) + U'(t) - 12U(t) = 0$

Introduction to the Problem

The initial value problem (IVP) presented:

$$U''(t) + U'(t) - 12\, U(t) = 0$$

with initial conditions:

$$U(0) = 70, \quad U'(0) = 18$$

is a second-order linear homogeneous differential equation with constant coefficients. Such equations frequently appear in physics, engineering, and applied mathematics, modeling systems such as mechanical oscillations, electrical circuits, and thermal processes.

The goal is to solve this IVP, which involves:

- Deriving the general solution to the differential equation.
- Applying initial conditions to determine specific solution constants.
- Showing that the solution takes the form $U(t) = c e^{\lambda t}$, and examining under what circumstances this form is valid.

This detailed exploration will elaborate on solving the differential equation step by step, analyzing the characteristic equation, characteristic roots, the general solution, and interpretation of the specific solution form.

Understanding the Differential Equation

Type of Equation

The given differential equation:

$$U''(t) + U'(t) - 12 U(t) = 0$$

is a second-order linear homogeneous differential equation with constant coefficients. Its linearity and constant coefficients make it amenable to characteristic equation methods.

Physical and Mathematical Context

- Such equations often model systems where the current state and its rate of change influence the acceleration or second derivative.
- The coefficients (1 for U' and -12 for U) determine the behavior of solutions—whether oscillatory, exponential decay, or growth.

Methodology for Solving the IVP

Step 1: Write the characteristic equation

Substituting a trial solution of the form $U(t) = e^{rt}$:

$$
U(t) = e^{rt} \quad \Rightarrow \quad U'(t) = r e^{rt}, \quad U''(t) = r^2 e^{rt}
$$

Plug into the differential equation:

$$
r^2 e^{rt} + r e^{rt} - 12 e^{rt} = 0
$$

Dividing through by $e^{rt} \neq 0$:

$$
r^2 + r - 12 = 0
$$

This quadratic equation is known as the characteristic equation:

$$
r^2 + r - 12 = 0
$$

Step 2: Solve the characteristic equation

Factoring or applying quadratic formula:

$$
r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
$$

where $a=1$, $b=1$, $c=-12$.

Calculate discriminant:

$$
\Delta = 1^2 - 4 \times 1 \times (-12) = 1 + 48 = 49
$$

Find roots:

$$
r = \frac{-1 \pm \sqrt{49}}{2} = \frac{-1 \pm 7}{2}
$$

Thus,

- $r_1 = \frac{-1 + 7}{2} = \frac{6}{2} = 3$
- $r_2 = \frac{-1 - 7}{2} = \frac{-8}{2} = -4$

Step 3: Write the general solution

Since roots are real and distinct, the general solution is:

$$
U(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} = C_1 e^{3t} + C_2 e^{-4t}
$$

where C_1 and C_2 are constants determined by initial conditions.

Applying Initial Conditions

Step 4: Use initial conditions to find C_1 and C_2

Given:

$$

$$
U(0) = 70 \quad \Rightarrow \quad C_1 e^{0} + C_2 e^{0} = C_1 + C_2 = 70
$$

$$
U'(t) = 3 C_1 e^{3t} - 4 C_2 e^{-4t}
$$

At $t=0$:

$$
U'(0) = 3 C_1 - 4 C_2 = 18
$$

This yields a system of equations:

$$
\begin{cases}
C_1 + C_2 = 70 \\
3 C_1 - 4 C_2 = 18
\end{cases}
$$

Solve for C_1 and C_2:

From the first:

$$
C_2 = 70 - C_1
$$

Substitute into second:

$$
3 C_1 - 4 (70 - C_1) = 18
$$

$$
3 C_1 - 280 + 4 C_1 = 18
$$

$$
7 C_1 = 298
$$

$$
C_1 = \frac{298}{7} \approx 42.5714
$$

Calculate C_2:

$$
C_2 = 70 - \frac{298}{7} = \frac{490}{7} - \frac{298}{7} = \frac{192}{7} \approx 27.4286
$$

Explicit Solution to the IVP

Putting everything together:

$$
U(t) = \frac{298}{7} e^{3t} + \frac{192}{7} e^{-4t}
$$

This explicit form satisfies both the differential equation and initial conditions.

Showing the Solution in the Form $U(t) = c e^{\lambda t}$

Analysis of the Exponential Solution Form

The problem statement suggests demonstrating $U(t) = c e^{\lambda t}$, which corresponds to solutions when the differential equation admits single exponential solutions—typically in the case of repeated roots or specific eigenvalues.

However, since the characteristic roots are distinct (3 and -4), the general solution is a linear combination of two exponentials.

When does $U(t) = c e^{\lambda t}$ hold?

- When the differential equation has repeated roots, i.e., $r_1 = r_2$.
- When the solution is specifically a single exponential, which occurs if initial conditions are chosen so that the other exponential term vanishes.

Example: Special Cases

- If initial conditions satisfy conditions such that $C_2 = 0$, the solution simplifies to:

$$
U(t) = C_1 e^{3t}
$$

- Similarly, if $C_1 = 0$, then:

$$
U(t) = C_2 e^{-4t}
$$

In such cases, the solution indeed takes the form $U(t) = c e^{\lambda t}$.

Deep Dive into the Solution Structure

Stability and Behavior of the Solution

- The exponentials e^{3t} and e^{-4t} dominate the long-term behavior.
- Since $3 > 0$, e^{3t} grows exponentially as $t \to \infty$.
- Conversely, e^{-4t} decays exponentially.

The initial conditions determine whether the solution exhibits growth, decay, or a combination thereof.

Physical Interpretation

- The exponential growth term could model an unstable system, such as an amplifier with positive feedback.
- The decay term could represent damping or stabilization effects.
- The constants C_1, C_2 encode initial energy, displacement, or velocity depending on the physical context.

Additional Considerations and Advanced Topics

Homogeneous vs. Nonhomogeneous Equations

- The current problem is homogeneous; solutions are linear combinations of exponentials.
- For nonhomogeneous cases, particular solutions are added to the general solution.

Superposition Principle

- The linearity of the differential equation allows for superposition:

$$
U(t) = C_1 e^{3t} + C_2 e^{-4t}
$$

- Superposition facilitates solving complex systems by decomposing into simpler parts.

Alternative Solution Methods

- Laplace Transform: Useful for initial value problems with nonhomogeneous terms or complex boundary conditions.
- Series Solutions: When coefficients are variable, power series methods become relevant.
- Numerical Methods: For more complicated equations or initial conditions, numerical integration (Euler, Runge-Kutta) is employed.

Conclusion and Summary

The differential equation:

\[
U''

Consider The Following Ivp U T U T 12u T 0 1 U 0 70 And U 0 18 Show That U T Ce

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