Consider The Function. F(x) = Sin(x), 0 < X < Find The Half-range Cosine Expansion Of The Given

Consider The Function. F(x) = Sin(x), 0 < X < π. Find The Half-range Cosine Expansion Of The Given

When exploring Fourier series and their applications, one of the fundamental tasks is to determine the half-range cosine expansion of a given function. In this article, we will analyze the function F(x) = sin(x) defined on the interval (0, π) and systematically derive its half-range cosine expansion. This process is essential in various fields such as heat conduction, vibrations, and signal processing, where functions are often expanded into series to facilitate analysis and solution of differential equations.

Understanding Half-Range Cosine Series

What is a Half-Range Cosine Series?

A half-range cosine series is a type of Fourier series expansion used to express a function defined on the interval (0, L) in terms of cosine functions. Unlike the full Fourier series, which extends over a symmetric interval around zero, the half-range series focuses only on the positive part of the domain, which makes it particularly useful in problems with boundary conditions specified on (0, L).

Mathematically, for a function f(x) defined on (0, L), the half-range cosine expansion is given by:

$$ f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos \left(\frac{n \pi x}{L} \right) $$

where the coefficients a_n are computed as:

$$ a_n = \frac{2}{L} \int_0^L f(x) \cos \left(\frac{n \pi x}{L} \right) dx $$

for $n = 0, 1, 2, \dots$.

Why Use the Half-Range Cosine Series?

This expansion is particularly useful when dealing with functions that are even extensions of the original function. It simplifies the process of solving boundary value problems with specific boundary conditions, especially in heat and wave equations.

Applying the Half-Range Cosine Expansion to F(x) = Sin(x)

Step 1: Define the Problem Parameters

Given:

- Function: $f(x) = \sin(x)$
- Interval: $0 < x < \pi$ (here, $L = \pi$)

Our goal is to find the half-range cosine series expansion of $\sin(x)$ over $(0, \pi)$.

Step 2: Compute the Coefficients a_n

Using the formula for a_n:

$$ a_n = \frac{2}{L} \int_0^L f(x) \cos \left(\frac{n \pi x}{L} \right) dx $$

Substituting $L = \pi$, the coefficients become:

$$ a_n = \frac{2}{\pi} \int_0^\pi \sin(x) \cos(n x) dx $$

Our task is to evaluate this integral for each $n \geq 0$.

Step 3: Evaluate the Integral $I_n = \int_0^\pi \sin(x) \cos(n x) dx$

Using trigonometric identities, we can simplify the integral:

- Recall that:

$$ \sin(x) \cos(n x) = \frac{1}{2} [\sin(x + n x) + \sin(x - n x)] $$

$$ = \frac{1}{2} [\sin((n + 1) x) + \sin((1 - n) x)] $$

Thus,

$$ I_n = \frac{1}{2} \int_0^\pi \left[\sin((n + 1) x) + \sin((1 - n) x) \right] dx $$

Now, evaluate each integral separately.

- For $\int_0^\pi \sin(k x) dx$:

$$ \int_0^\pi \sin(k x) dx = \left[-\frac{\cos(k x)}{k} \right]_0^\pi = -\frac{\cos(k \pi)}{k} + \frac{\cos(0)}{k} = \frac{1 - (-1)^k}{k} $$

since $\cos(k \pi) = (-1)^k$.

Now, compute I_n:

$$I_n = \frac{1}{2} \left[\frac{1 - (-1)^{n+1}}{n+1} + \frac{1 - (-1)^{1 - n}}{1 - n} \right]$$

But note that:

- For $n \geq 0$, $(-1)^{n+1} = -(-1)^n$, so

$$1 - (-1)^{n+1} = 1 + (-1)^n$$

Similarly,

$$(-1)^{1 - n} = (-1)^1 \cdot (-1)^{-n} = -(-1)^n$$

thus,

$$1 - (-1)^{1 - n} = 1 + (-1)^n$$

Therefore,

$$I_n = \frac{1}{2} \left[\frac{1 + (-1)^n}{n+1} + \frac{1 + (-1)^n}{1 - n} \right]$$

Case 1: n even

If n is even, $(-1)^n = 1$, so:

$$I_n = \frac{1}{2} \left[\frac{2}{n+1} + \frac{2}{1 - n} \right] = \frac{1}{2} \left(\frac{2}{n+1} + \frac{2}{1 - n} \right)$$

Simplify:

$$I_n = \frac{1}{2} \times 2 \left(\frac{1}{n+1} + \frac{1}{1 - n} \right) = \left(\frac{1}{n+1} + \frac{1}{1 - n} \right)$$

Notice $1 - n = -(n - 1)$, so:

$$I_n = \frac{1}{n+1} - \frac{1}{n - 1}$$

Case 2: n odd

If n is odd, $(-1)^n = -1$, so:

$$I_n = \frac{1}{2} \left[\frac{0}{n+1} + \frac{0}{1 - n} \right] = 0$$

Thus, the integral I_n is zero for odd n.

Summary:

- For even n:

$$ I_n = \frac{1}{n+1} - \frac{1}{n - 1} $$

- For odd n:

$$ I_n = 0 $$

Step 4: Compute the Coefficients a_n

Recall:

$$ a_n = \frac{2}{\pi} I_n $$

Therefore,

- For even n:

$$ a_n = \frac{2}{\pi} \left(\frac{1}{n+1} - \frac{1}{n - 1} \right) $$

- For odd n:

$$ a_n = 0 $$

And note that for $n = 0$:

$$ a_0 = \frac{2}{\pi} \int_0^\pi \sin(x) dx = \frac{2}{\pi} \left[-\cos(x) \right]_0^\pi = \frac{2}{\pi} (-\cos \pi + \cos 0) = \frac{2}{\pi} (-(-1) + 1) = \frac{2}{\pi} (1 + 1) = \frac{4}{\pi} $$

So,

$$ a_0 = \frac{4}{\pi} $$

Final Expression of the Half-Range Cosine Series

Putting all the parts together, the half-range cosine expansion of $f(x) = \sin(x)$ on $(0, \pi)$ is:

$$ \sin(x) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos(n x) $$

where:

- $a_0 = \frac{4}{\pi}$,
- For even n:

$$ a_n = \frac{2}{\pi} \left(\frac{1}{n+1} - \frac{1}{n - 1} \right) $$,

- For odd \(n \):

\[a_n = 0 \].

Thus,

\[\boxed{
\sin(x) = \frac{2}{\pi} + \sum_{\substack{n=2 \\ n \text{ even}}}^\infty \frac{2}{\pi}
\left(\frac{1}{

Frequently Asked Questions

What is the main goal when finding the half-range cosine expansion of the function $F(x) = \sin(x)$ over the interval $0 < x < \pi$?

The goal is to express the function as a sum of cosine terms that approximate it within the interval, specifically using half-range Fourier cosine series to analyze or reconstruct the function efficiently.

How is the half-range cosine expansion of $F(x) = \sin(x)$ derived for $0 < x < \pi$?

It is derived by computing the cosine Fourier coefficients over the interval $(0, \pi)$, then expressing $F(x)$ as a series sum of these cosine terms multiplied by the coefficients.

What is the general formula for the cosine Fourier series expansion of a function defined on $0 < x < \pi$?

The general formula is $F(x) = a_0/2 + \sum_{n=1}^\infty a_n \cos(n x)$, where a_n are the Fourier cosine coefficients calculated as $a_n = (2/\pi) \int_0^\pi F(x) \cos(n x)\, dx$.

How do you compute the Fourier cosine coefficients for $F(x) = \sin(x)$?

You compute them using $a_n = (2/\pi) \int_0^\pi \sin(x) \cos(n x)\, dx$, which involves integrating the product of $\sin(x)$ and $\cos(n x)$ over the interval.

What is the value of the Fourier cosine coefficient a_0 for $F(x) = \sin(x)$?

Since $a_0 = (2/\pi) \int_0^\pi \sin(x)\, dx$, and $\int_0^\pi \sin(x)\, dx = 2$, thus $a_0 = (2/\pi)\, 2 = 4/\pi$.

What is the general form of the Fourier cosine series for sin(x) over $0 < x < \pi$?

It is expressed as $F(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos(n x)$, with coefficients a_n computed via integrals involving sin(x) and cos(n x).

Are there any symmetry properties of sin(x) that simplify the half-range cosine expansion process?

Yes, since sin(x) is an odd function about $x = \pi/2$, but over $(0, \pi)$, it is not symmetric about the origin, so the expansion uses cosine terms which are even functions; this symmetry helps separate the series components.

Can the half-range cosine expansion of sin(x) be expressed in a closed form, or is it an infinite series?

It is generally expressed as an infinite series, but for practical purposes, it can be approximated by truncating the series after a finite number of terms.

What applications can benefit from the half-range cosine expansion of functions like sin(x)?

Applications include solving differential equations, signal processing, heat transfer problems, and approximation of functions within a specified interval for engineering and physics problems.

How does the choice of the interval (0, π) influence the form of the cosine expansion for sin(x)?

Choosing the interval $(0, \pi)$ ensures the expansion uses cosine terms that satisfy the boundary conditions at the endpoints, allowing for a suitable half-range Fourier cosine series representation of sin(x).

Additional Resources

Consider The Function. F(x) = Sin(x), 0 < X < π. Find The Half-range Cosine Expansion Of The Given

When delving into the world of Fourier series and their applications, understanding the half-range cosine expansion of functions becomes essential, especially for functions defined on a specific interval such as $0 < x < \pi$. In this context, we are asked to consider the function $f(x) = \sin(x)$ on the interval $0 < x < \pi$, and find its half-range cosine expansion. This process involves expressing $f(x)$ as an infinite sum of cosine functions tailored to the half-interval, which has numerous applications in solving boundary value problems, signal processing, and mathematical physics.

Understanding the Half-Range Cosine Series

Before diving into the specific problem, it is crucial to understand what a half-range cosine series is and how it differs from the full Fourier cosine series.

What is a Half-Range Cosine Series?

A half-range cosine series is a Fourier cosine series expansion of a function defined on the interval $0 < x < L$, where the function is extended to $[0, L]$ as an even function. The expansion allows us to approximate the function within this interval using only cosine terms, which are inherently even functions.

Mathematically, if $f(x)$ is defined on $[0, L]$, its half-range cosine expansion is:

$$
f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n \pi x}{L}\right)
$$

where the coefficients a_n are given by:

$$
a_0 = \frac{2}{L} \int_0^L f(x) \, dx
$$

$$
a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n \pi x}{L}\right) dx, \quad n \geq 1
$$

Setting Up the Problem: $f(x) = \sin(x)$ on $0 < x < \pi$

Given the function $f(x) = \sin(x)$, where $0 < x < \pi$, our goal is to find its half-range cosine expansion on this interval. Since the interval is $0 < x < \pi$, we identify $L=\pi$.

Step 1: Recognize the Nature of $f(x)$

- The sine function is inherently odd over $[-\pi, \pi]$, but when considering the interval $[0, \pi]$, $\sin(x)$ is positive and continuous.
- The half-range cosine expansion requires the function to be extended as an even function about $x=0$. For the purpose of expansion, we consider the even extension of $f(x)$ about $x=0$.

Step 2: Determine the Coefficients

Using the formulas for the coefficients:

$$
a_0 = \frac{2}{\pi} \int_0^{\pi} \sin(x) \, dx
$$

\[
a_n = \frac{2}{\pi} \int_0^{\pi} \sin(x) \cos(n x) \, dx
\]

for \(n=1, 2, 3, \dots\).

Computing the Coefficients

Calculating \(a_0\)

\[
a_0 = \frac{2}{\pi} \int_0^{\pi} \sin(x) \, dx
\]

The integral:

\[
\int_0^{\pi} \sin(x) \, dx = [-\cos(x)]_0^{\pi} = (-\cos \pi) - (-\cos 0) = (-(-1)) - (-1) = 1 + 1 = 2
\]

Thus,

\[
a_0 = \frac{2}{\pi} \times 2 = \frac{4}{\pi}
\]

Calculating \(a_n\)

\[
a_n = \frac{2}{\pi} \int_0^{\pi} \sin(x) \cos(n x) \, dx
\]

To evaluate this integral, use the product-to-sum identity:

\[
\sin(x) \cos(n x) = \frac{1}{2} [\sin((n+1) x) + \sin((n-1) x)]
\]

Therefore,

\[
a_n = \frac{2}{\pi} \times \frac{1}{2} \int_0^{\pi} [\sin((n+1) x) + \sin((n-1) x)] dx
\]

Simplify:

\[
a_n = \frac{1}{\pi} \left(\int_0^{\pi} \sin((n+1) x) dx + \int_0^{\pi} \sin((n-1) x) dx

\right)
\]

Compute each integral separately:

\[
\int_0^{\pi} \sin(k x) dx = \left[-\frac{\cos(k x)}{k} \right]_0^{\pi} = -\frac{\cos(k \pi)}{k} + \frac{\cos(0)}{k} = \frac{1 - \cos(k \pi)}{k}
\]

Thus,

\[
a_n = \frac{1}{\pi} \left(\frac{1 - \cos((n+1) \pi)}{n+1} + \frac{1 - \cos((n-1) \pi)}{n-1} \right)
\]

Recall that:

\[
\cos(m \pi) = (-1)^m
\]

So,

\[
a_n = \frac{1}{\pi} \left(\frac{1 - (-1)^{n+1}}{n+1} + \frac{1 - (-1)^{n-1}}{n-1} \right)
\]

Simplifying the Coefficients

Behavior of \(\cos((n \pm 1) \pi)\)

- When \(n\) is even:

- \(n\) even \(\Rightarrow n+1\) odd, \(n-1\) odd.

- \(\cos(\text{odd} \times \pi) = -1\).

- When \(n\) is odd:

- \(n\) odd \(\Rightarrow n+1\) even, \(n-1\) even.

- \(\cos(\text{even} \times \pi) = 1\).

Final expression for \(a_n\):

- For even \(n\):

\[
a_n = \frac{1}{\pi} \left(\frac{1 - (-1)^{n+1}}{n+1} + \frac{1 - (-1)^{n-1}}{n-1} \right)
\]

Since \(n\) even:

- \(n+1\) odd \(\Rightarrow (-1)^{n+1} = -1\),
- \(n-1\) odd \(\Rightarrow (-1)^{n-1} = -1\).

Plug in:

\[
a_n = \frac{1}{\pi} \left(\frac{1 - (-1)}{n+1} + \frac{1 - (-1)}{n-1} \right) =
\frac{1}{\pi} \left(\frac{2}{n+1} + \frac{2}{n-1} \right)
\]

- For odd \(n\):

- \(n+1\) even \(\Rightarrow (-1)^{n+1} = 1\),
- \(n-1\) even \(\Rightarrow (-1)^{n-1} = 1\).

Plug in:

\[
a_n = \frac{1}{\pi} \left(\frac{1 - 1}{n+1} + \frac{1 - 1}{n-1} \right) = 0
\]

Final Expression for the Half-Range Cosine Series

Summarizing:

\[
a_0 = \frac{4}{\pi}
\]

\[
a_n =
\begin{cases}
\displaystyle \frac{2}{\pi} \left(\frac{1}{n+1} + \frac{1}{n-1} \right), & \text{if } n
\text{ is even} \\
0, & \text{if } n \text{ is odd}
\end{cases}
\]

Thus, the half-range cosine expansion of \(f(x) = \sin(x)\) on \([0, \pi]\) is:

\[
\boxed{
f(x) = \frac{a_0}{2} + \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} a_n \cos(n x)

}
\]

which explicitly becomes:

\[
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