Consider The Second Order Differential Equation With Initial Conditions U" - 4u' 5.5u = -4.5 Sin(3t),

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Understanding second-order differential equations is fundamental in many fields such as physics, engineering, and applied mathematics. This particular differential equation, which involves second derivatives, first derivatives, and a sinusoidal forcing term, presents a common scenario where the behavior of dynamic systems can be analyzed and predicted. In this article, we will explore the solution methods, initial conditions, and practical applications of the differential equation:

$$u'' - 4u' + 5.5u = -4.5 \sin(3t)$$

along with initial conditions, typically specified as $u(0)$ and $u'(0)$.

Understanding the Structure of the Differential Equation

General Form of Second-Order Linear Differential Equations

The given differential equation is a linear, nonhomogeneous second-order differential equation. It can be generally written as:

$$u'' + p(t) u' + q(t) u = r(t)$$

where:

- u'' is the second derivative of $u(t)$ with respect to t,
- $p(t)$ and $q(t)$ are functions of t, often constants in many physical problems,
- $r(t)$ is the forcing function or nonhomogeneous term.

In our specific case:

$$u'' - 4u' + 5.5u = -4.5 \sin(3t)$$

which simplifies to constant coefficients:

$$u'' + (-4) u' + 5.5 u = -4.5 \sin(3t)$$

This form indicates the equation models a system with damping (from the u' term), stiffness or restoring force (from the u term), and external forcing (the sinusoidal term).

Homogeneous Solution: Solving the Associated Homogeneous Equation

Forming the Characteristic Equation

The first step in solving such equations involves finding the complementary (homogeneous) solution, which solves:

$$u'' - 4u' + 5.5u = 0$$

The characteristic equation associated with this differential equation is:

$$r^2 - 4r + 5.5 = 0$$

This quadratic can be solved using the quadratic formula:

$$r = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 5.5}}{2}$$

Calculating discriminant:

$$\Delta = 16 - 22 = -6$$

Since the discriminant is negative, roots are complex conjugates:

$$r = \frac{4 \pm i \sqrt{6}}{2} = 2 \pm i \frac{\sqrt{6}}{2}$$

which simplifies to:

$$r = 2 \pm i \frac{\sqrt{6}}{2}$$

Homogeneous Solution

The general homogeneous solution, based on roots $r = \alpha \pm i \beta$, is:

$$u_h(t) = e^{\alpha t} \left(C_1 \cos(\beta t) + C_2 \sin(\beta t) \right)$$

where:

- $\alpha = 2$,

- $\beta = \frac{\sqrt{6}}{2}$.

Thus,

$$u_h(t) = e^{2t} \left(C_1 \cos \left(\frac{\sqrt{6}}{2} t \right) + C_2 \sin \left(\frac{\sqrt{6}}{2} t \right) \right)$$

This solution describes the natural response of the system — how it evolves without external forcing, influenced only by initial conditions.

Particular Solution: Addressing the Forcing Term

Method of Undetermined Coefficients

Since the nonhomogeneous term is $-4.5 \sin(3t)$, a sinusoidal forcing, the method of undetermined coefficients suggests trying a particular solution of the form:

$$u_p(t) = A \sin(3t) + B \cos(3t)$$

where A and B are constants to be found.

Calculating Derivatives

To substitute into the differential equation, we need derivatives:

$$u_p' = 3A \cos(3t) - 3B \sin(3t)$$
$$u_p'' = -9A \sin(3t) - 9B \cos(3t)$$

Substituting into the Differential Equation

Plugging u_p, u_p', and u_p'' into the original equation:

$$
(-9A \sin(3t) - 9B \cos(3t)) - 4(3A \cos(3t) - 3B \sin(3t)) + 5.5 (A \sin(3t) + B \cos(3t)) = -4.5 \sin(3t)
$$

Simplify term-by-term:

$$
-9A \sin(3t) - 9B \cos(3t) - 12A \cos(3t) + 12B \sin(3t) + 5.5A \sin(3t) + 5.5B \cos(3t) = -4.5 \sin(3t)
$$

Group $\sin(3t)$ and $\cos(3t)$ terms:

$$
[\text{sin terms}]: (-9A + 12B + 5.5A) \sin(3t) = ((-9A + 5.5A) + 12B) \sin(3t) = (-3.5A + 12B) \sin(3t)
$$

$$
[\text{cos terms}]: (-9B - 12A + 5.5B) \cos(3t) = ((-9B + 5.5B) - 12A) \cos(3t) = (-3.5B - 12A) \cos(3t)
$$

Set equal to the right side:

$$
(-3.5A + 12B) \sin(3t) + (-3.5B - 12A) \cos(3t) = -4.5 \sin(3t)
$$

Matching coefficients:

$$
-3.5A + 12B = -4.5
$$
$$
-3.5B - 12A = 0
$$

This system can be solved for A and B.

Solving for A and B

1. From the second equation:

   $$ -3.5B - 12A = 0 \Rightarrow 3.5B = -12A \Rightarrow B = -\frac{12A}{3.5} = -\frac{24A}{7} $$

2. Substitute B into the first equation:

   $$ -3.5A + 12 \left(-\frac{24A}{7} \right) = -4.5 $$

   Simplify:

   $$ -3.5A - \frac{288A}{7} = -4.5 $$

   Express all with denominator 7:

$$ -\frac{24.5A}{7} - \frac{288A}{7} = -4.5 $$

Combine:

$$ -\frac{24.5A + 288A}{7} = -4.5 $$
$$ -\frac{312.5A}{7} = -4.5 $$

Multiply both sides by 7:

$$ -312.5A = -4.5 \times 7 = -31.5 $$

Divide:

$$ A = \frac{-31.5}{-312.5} = \frac{31.5}{312.5} $$

Simplify numerator and denominator:

$$ A = \frac{63/2}{625/2} = \frac{63}{2} \times \frac{2}{625} = \frac{63}{625} $$

Calculate B:

$$ B = -\frac{24A}{7} = -\frac{24 \times 63/625}{7} = -\frac{1512/625}{7} = -\frac{1512}{625 \times 7} = -\frac{1512}{4375} $$

Final Particular Solution and General Solution

The particular solution is:

$$ u_p(t) = A \sin(3t) + B \cos(3t) = \frac{63}{625} \sin(3t) - \frac{1512}{4375} \cos(3t) $$

Thus, the general solution to the differential equation is:

\[u(t) = u_h(t

Frequently Asked Questions

What is the standard form of the differential equation given as U'' - 4U' + 5.5U = -4.5 sin(3t)?

The standard form is already given: U'' - 4U' + 5.5U = -4.5 sin(3t). It is a second-order linear nonhomogeneous differential equation with constant coefficients and a sinusoidal forcing term.

How do you find the general solution of the homogeneous equation U'' - 4U' + 5.5U = 0?

To find the homogeneous solution, solve the characteristic equation $r^2 - 4r + 5.5 = 0$. The roots determine the form of the homogeneous solution, which can be real or complex conjugates depending on the discriminant.

What method is suitable for solving the particular solution of U'' - 4U' + 5.5U = -4.5 sin(3t)?

The method of undetermined coefficients is suitable here, assuming a particular solution of the form $A \sin(3t) + B \cos(3t)$, and solving for A and B.

How do initial conditions influence the solution of the differential equation?

Initial conditions allow us to determine the specific values of constants in the general solution, resulting in a particular solution that satisfies the initial state of the system.

What is the physical interpretation of this differential equation in modeling systems?

This second-order differential equation can model a damped oscillator with external sinusoidal forcing, such as a mass-spring system with damping subjected to a periodic force.

What are the steps to solve for the particular solution in this differential equation?

First, find the homogeneous solution by solving the characteristic equation. Then, assume a particular solution form based on the forcing function, substitute into the original equation, and solve for the coefficients A and B. Finally, combine solutions and apply initial conditions if provided.

Additional Resources

Consider The Second Order Differential Equation With Initial Conditions U" - 4u' 5.5u = -4.5 Sin(3t)

The study of differential equations is a cornerstone of mathematical analysis, providing essential tools for modeling a wide array of physical, biological, and engineering systems. Among these, second-order differential equations are particularly prominent due to their capacity to describe dynamic phenomena involving acceleration, oscillations, and wave propagation. In this article, we delve into the intricacies of a specific second-order differential equation:

U" - 4u' + 5.5u = -4.5 sin(3t)

along with its initial conditions. This comprehensive exploration aims to elucidate its structure, solution methods, and the physical or theoretical significance underlying its form.

Understanding the Differential Equation: Structure and Components

What Is a Second-Order Differential Equation?

A second-order differential equation involves the second derivative of an unknown function, typically denoted as u'' or U'', along with functions of the independent variable (here, t) and possibly the unknown function itself. These equations often model systems where acceleration or curvature plays a vital role, such as pendulums, mechanical vibrations, electrical circuits, and more.

The Equation at a Glance

The given differential equation is:

$$U'' - 4u' + 5.5u = -4.5 \sin(3t)$$

with initial conditions (though unspecified in the prompt, typically):

$$u(0) = u_0, \quad u'(0) = u_1$$

The notation suggests:

- u'' (or U'') — the second derivative of $u(t)$.
- u' — the first derivative of $u(t)$.
- $u(t)$ — the unknown function of t.

This form indicates a linear nonhomogeneous differential equation with constant coefficients and a sinusoidal forcing term.

Components Breakdown

- Homogeneous Part: $u'' - 4u' + 5.5u = 0$
- Particular Solution: To account for the right-hand side $-4.5 \sin(3t)$
- Initial Conditions: Necessary to determine the specific solution satisfying the system's initial state.

Homogeneous Equation Analysis

Solving the Homogeneous Equation

The homogeneous differential equation associated with the given problem is:

$$u'' - 4u' + 5.5u = 0$$

The standard approach involves finding the characteristic equation:

$$r^2 - 4r + 5.5 = 0$$

which yields roots determining the general homogeneous solution.

Characteristic Equation and Roots

Calculating roots:

$$r = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 5.5}}{2}$$
$$r = \frac{4 \pm \sqrt{16 - 22}}{2}$$
$$r = \frac{4 \pm \sqrt{-6}}{2}$$
$$r = \frac{4 \pm i \sqrt{6}}{2}$$
$$r = 2 \pm i \frac{\sqrt{6}}{2}$$

Thus, the roots are complex conjugates:

$$r_1 = 2 + i \frac{\sqrt{6}}{2}$$
$$r_2 = 2 - i \frac{\sqrt{6}}{2}$$

Homogeneous Solution

Given the roots, the general homogeneous solution is:

$$u_h(t) = e^{2t} \left[C_1 \cos \left(\frac{\sqrt{6}}{2} t \right) + C_2 \sin \left(\frac{\sqrt{6}}{2} t \right) \right]$$

where C_1 and C_2 are constants determined by initial conditions.

Physical Interpretation

This solution describes oscillatory behavior modulated by exponential growth, indicating a system with an inherent damping or amplification depending on the context. The imaginary parts of roots reflect oscillatory components, while the real parts contribute to exponential behavior.

Particular Solution: Addressing the Forcing Term

The Forcing Function

The right-hand side is:

$$ -4.5 \sin(3t) $$

which is a sinusoidal forcing term. To find a particular solution, methods like undetermined coefficients or variation of parameters are appropriate.

Method 1: Undetermined Coefficients

Given the forcing term's sinusoidal form, assume a particular solution of the form:

$$ u_p(t) = A \sin(3t) + B \cos(3t) $$

Substituting $u_p(t)$ into the differential equation and solving for A and B allows us to find the particular solution.

Derivation of Coefficients

Calculations involve:

- Computing derivatives:

$$ u_p' = 3A \cos(3t) - 3B \sin(3t) $$
$$ u_p'' = -9A \sin(3t) - 9B \cos(3t) $$

- Substituting into the original equation:

$$ (-9A \sin(3t) - 9B \cos(3t)) - 4 (3A \cos(3t) - 3B \sin(3t)) + 5.5 (A \sin(3t) + B \cos(3t)) = -4.5 \sin(3t) $$

- Simplifying:

$$ -9A \sin(3t) - 9B \cos(3t) - 12A \cos(3t) + 12B \sin(3t) + 5.5A \sin(3t) + 5.5B \cos(3t) = -4.5 \sin(3t) $$

- Group coefficients:

$$
\left(-9A + 12B + 5.5A \right) \sin(3t) + \left(-9B - 12A + 5.5B \right) \cos(3t) = -4.5 \sin(3t)
$$

which simplifies to:

$$
(-3.5A + 12B) \sin(3t) + (-12A - 3.5B) \cos(3t) = -4.5 \sin(3t)
$$

Matching coefficients:

$$
-3.5A + 12B = -4.5
$$
$$
-12A - 3.5B = 0
$$

Solving for A and B:

From the second equation:

$$
-12A = 3.5B \Rightarrow A = -\frac{3.5}{12} B = -\frac{7}{24} B
$$

Substitute into the first:

$$
-3.5 \left(-\frac{7}{24} B \right) + 12 B = -4.5
$$
$$
\frac{3.5 \times 7}{24} B + 12 B = -4.5
$$
$$
\frac{24.5}{24} B + 12 B = -4.5
$$
$$
\left(1.0208 + 12 \right) B = -4.5
$$
$$
13.0208 B = -4.5
$$
$$
B = -\frac{4.5}{13.0208} \approx -0.3457
$$

Then,

$$
A = -\frac{7}{24} \times (-0.3457) \approx 0.1008
$$

Final Particular Solution

$$
u_p(t) \approx 0.1008 \sin(3t) - 0.3457 \cos(3t)
$$

Complete General Solution

Combining homogeneous and particular solutions:

$$
u(t) = e^{2t} \left[C_1 \cos \left(\frac{\sqrt{6}}{2} t \right) + C_2 \sin \left(\frac{\sqrt{6}}{2} t \right) \right] + 0.1008 \sin(3t) - 0.3457 \cos(3t)
$$

Initial conditions (not specified) would allow solving for C_1 and C_2.

Physical and Mathematical Implications

Oscillatory Dynamics and Resonance

The presence of sinusoidal forcing terms coupled with complex conjugate roots in the homogeneous solution suggests the system exhibits oscillations with exponential modulation—either growth or decay depending on the real parts of roots. When the forcing frequency approaches the natural frequency of the system, resonance phenomena can occur, potentially amplifying oscillations significantly.

Damping or Amplification

The exponential term e^{2t} indicates that, without damping, solutions tend to grow exponentially, implying an unstable system unless additional damping mechanisms are present. The coefficient $-4u'$ in the original differential equation acts as a damping or amplification term, which, depending on its

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