Consider The Sequence (an) Given By A1 = 1. A2 = 2, An+1= 1/2(an+an-1) For N > 2.We Will Show That

Consider The Sequence (an) Given By A1 = 1. A2 = 2, An+1= 1/2(an+an-1) For N > 2. We Will Show That

---

Introduction

Sequences are fundamental objects in mathematics, serving as building blocks for understanding limits, series, and various applications across science and engineering. Among the myriad types of sequences, recursive sequences—where each term is defined in terms of previous terms—are particularly intriguing due to their self-referential nature.

In this article, we explore a specific recursive sequence defined as follows:

- Initial terms: $A_1 = 1$, $A_2 = 2$
- Recursive relation: for $n > 2$,

$$
A_{n+1} = \frac{1}{2}(A_n + A_{n-1})
$$

Our goal is to analyze this sequence thoroughly. We will investigate its behavior, convergence properties, limit, and whether it approaches a steady value as n increases. This exploration will involve deriving explicit formulas, studying the sequence's monotonicity, and confirming the existence of limits.

Understanding the Sequence Definition

The Recursive Formula

The recursive relation:

$$
A_{n+1} = \frac{1}{2}(A_n + A_{n-1})
$$

is a linear recurrence relation with constant coefficients. It states that each subsequent term is the average of the previous two terms. This type of relation often indicates smoothing or averaging behavior, suggesting the sequence may stabilize over time.

Initial Conditions

- $A_1 = 1$
- $A_2 = 2$

Using these, subsequent terms can be calculated iteratively:

$$
A_3 = \frac{1}{2}(A_2 + A_1) = \frac{1}{2}(2 + 1) = 1.5
$$

$$
A_4 = \frac{1}{2}(A_3 + A_2) = \frac{1}{2}(1.5 + 2) = 1.75
$$

$$
A_5 = \frac{1}{2}(A_4 + A_3) = \frac{1}{2}(1.75 + 1.5) = 1.625
$$

As we continue calculating, we observe the sequence oscillates and appears to approach a certain value.

Analyzing the Behavior of the Sequence

Numerical Computation of Terms

Let's compute the first few terms to observe the pattern:

n	A_n
1	1
2	2
3	1.5
4	1.75
5	1.625
6	1.6875
7	1.65625
8	1.671875
9	1.6640625
10	1.66796875

The sequence oscillates around approximately 1.6667 and seems to be converging toward that value.

Intuition about the Limit

The pattern suggests the sequence converges to a limit L. Given the recursive relation, if $A_n \to L$ as $n \to \infty$, then both A_{n} and A_{n-1} tend to L.

Plugging into the recurrence:

$$
L = \frac{1}{2}(L + L) = \frac{1}{2} \times 2L = L
$$

which is consistent but does not determine L. To find L, we need additional insight.

Deriving the Limit and Explicit Formula

Solving the Recurrence Relation

The recurrence relation:

$$
A_{n+1} = \frac{1}{2}(A_n + A_{n-1})
$$

is linear and homogeneous. To solve it, we consider the associated characteristic equation.

Step 1: Homogeneous Equation

Write the recurrence as:

$$
A_{n+1} - \frac{1}{2}A_n - \frac{1}{2}A_{n-1} = 0
$$

Assuming a solution of the form $A_n = r^n$:

$$
r^{n+1} - \frac{1}{2} r^n - \frac{1}{2} r^{n-1} = 0
$$

Divide through by r^{n-1}:

$$
r^2 - \frac{1}{2} r - \frac{1}{2} = 0
$$

This quadratic equation:

$$
r^2 - \frac{1}{2} r - \frac{1}{2} = 0
$$

can be solved using the quadratic formula:

$$
r = \frac{\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - 4 \times 1 \times -\frac{1}{2}}}{2}
$$

Simplify inside the square root:

$$
\left(\frac{1}{2}\right)^2 - 4 \times 1 \times -\frac{1}{2} = \frac{1}{4} + 2 = \frac{1}{4} + 2 = \frac{1 + 8}{4} = \frac{9}{4}
$$

Thus,

$$
r = \frac{\frac{1}{2} \pm \sqrt{\frac{9}{4}}}{2} = \frac{\frac{1}{2} \pm \frac{3}{2}}{2}
$$

Compute both roots:

- For the positive root:

$$
r = \frac{\frac{1}{2} + \frac{3}{2}}{2} = \frac{2}{2} = 1
$$

- For the negative root:

$$
r = \frac{\frac{1}{2} - \frac{3}{2}}{2} = \frac{-1}{2} = -\frac{1}{2}
$$

Step 2: General Solution

The general solution to the recurrence is:

$$
A_n = \alpha \times 1^n + \beta \times \left(-\frac{1}{2}\right)^n = \alpha + \beta \left(-\frac{1}{2}\right)^n
$$

where α, β are constants determined by initial conditions.

Determining the Constants α and β

Use initial conditions:

$$
A_1 = 1 = \alpha + \beta \left(-\frac{1}{2}\right)^1 = \alpha - \frac{1}{2}\beta
$$
$$
A_2 = 2 = \alpha + \beta \left(-\frac{1}{2}\right)^2 = \alpha + \frac{1}{4}\beta
$$

This gives us a system of equations:

$$
\begin{cases}
1 = \alpha - \frac{1}{2}\beta \quad (1) \\
2 = \alpha + \frac{1}{4}\beta \quad (2)
\end{cases}
$$

Subtract (1) from (2):

$$
(2) - (1): \quad 2 - 1 = (\alpha - \alpha) + \left(\frac{1}{4}\beta + \frac{1}{2}\beta\right) \\
1 = \frac{3}{4}\beta
$$

So,

$$
\beta = \frac{4}{3}
$$

Substitute into (1):

$$
1 = \alpha - \frac{1}{2} \times \frac{4}{3} = \alpha - \frac{2}{3}
$$
$$
\alpha = 1 + \frac{2}{3} = \frac{3}{3} + \frac{2}{3} = \frac{5}{3}
$$

Explicit Formula for A_n

$$
\boxed{
A_n = \frac{5}{3} + \frac{4}{3} \left(-\frac{1}{2}\right)^n
}
$$

Limit of the Sequence

As $n \to \infty$:

$$
\left(-\frac{1}{2}\right)^n \to 0
$$

since $\left|-\frac{1}{2}\right| < 1$. Therefore,

$$
\boxed{
\lim_{n \to \infty} A_n = \frac{5}{3} \approx 1.6667
}
$$

which aligns with the numerical observations.

Properties of the Sequence

Monotonicity and Oscillation

The sequence oscillates because of the $\left(-\frac{1}{2}\right)^n$ term, which alternates sign. The magnitude diminishes over time, causing the sequence to approach $5/3$ from above and below, depending on the parity of n.

- For even \(n\

Frequently Asked Questions

What is the initial term of the sequence (an) given by A1 = 1 and A2 = 2?

The initial term of the sequence is A1 = 1.

How is each subsequent term (An+1) of the sequence defined?

Each subsequent term is defined by the recurrence relation An+1 = 1/2 (an + an-1) for n > 2.

What is the significance of analyzing the sequence (an) given by the recurrence relation?

Analyzing this sequence helps understand its long-term behavior, convergence, and potential limits based on the recursive definition.

Can we determine the limit of the sequence (an) as n approaches infinity?

Yes, by examining the recurrence relation, we can investigate whether the sequence converges to a specific value or diverges.

What methods are typically used to analyze the convergence of such recursive sequences?

Methods include solving the recurrence relation explicitly, using fixed-point analysis, or applying limit theorems to find the sequence's behavior as n becomes large.

What will be demonstrated in the upcoming proof regarding the sequence (an)?

The proof will show the behavior, convergence properties, or explicit formula for the sequence defined by the given recurrence relation.

Additional Resources

Consider The Sequence (an) Given By A1 = 1, A2 = 2, An+1 = 1/2 (an + an-1) for n > 2: A Deep Dive into Its Behavior and Convergence

---

Introduction

Sequences are fundamental objects of study in mathematics, serving as the building blocks for understanding limits, series, and the behavior of functions. Among these, recursive sequences—where each term is defined based on preceding terms—are especially intriguing due to their often complex and unexpected behaviors. In this detailed analysis, we explore a particular sequence defined by initial conditions and a recursive relation:

- A1 = 1
- A2 = 2
- An+1 = 1/2 (an + an-1), for n > 2

This sequence, which we will refer to as the "Consider The Sequence" (an), presents a compelling case for investigation, particularly concerning its convergence, limit behavior, and potential applications in mathematical modeling.

Background and Motivation

The recursive relation An+1 = 1/2 (an + an-1) can be viewed as a form of averaging process, reminiscent of methods used in numerical analysis, smoothing functions, or iterative algorithms. Understanding the long-term behavior of such a sequence is crucial, especially in contexts where stability and convergence determine the viability of iterative schemes.

The initial conditions A1=1 and A2=2 set the stage for examining how the sequence evolves, whether it stabilizes to a particular value, diverges, or exhibits oscillatory behavior. Given the nature of the recursive formula—essentially a weighted average—it is intuitive to suspect that the sequence might converge to some fixed point. The core objective is to rigorously establish this behavior and determine the explicit limit, if it exists.

Formal Definition and Basic Properties

Definition of the Sequence

Let (an) be a sequence defined as follows:

- $a_1 = 1$
- $a_2 = 2$
- For $n \geq 2$: $a_{n+1} = 1/2 (a_n + a_{n-1})$

This recursive relation can be viewed as a second-order linear recurrence with averaging weights.

Initial Terms and Pattern Observation

Calculating the first few terms provides insight:

- $a_1 = 1$
- $a_2 = 2$
- $a_3 = 1/2 (a_2 + a_1) = 1/2 (2 + 1) = 1.5$
- $a_4 = 1/2 (a_3 + a_2) = 1/2 (1.5 + 2) = 1.75$
- $a_5 = 1/2 (a_4 + a_3) = 1/2 (1.75 + 1.5) = 1.625$
- $a_6 = 1/2 (a_5 + a_4) = 1/2 (1.625 + 1.75) = 1.6875$

The sequence appears to be approaching a value near 1.75, then decreasing slightly, suggesting convergence toward a particular limit.

Mathematical Analysis: Convergence and Limit

Homogeneous Recurrence Relation

The recursive relation:

$$a_{n+1} = 1/2 (a_n + a_{n-1})$$

can be rewritten as:

$$a_{n+1} - 1/2\, a_n = 1/2\, a_{n-1}$$

This is a linear, second-order recurrence relation, which can be analyzed using characteristic equations.

Derivation of the Characteristic Equation

Suppose the sequence stabilizes to a limit L as $n \to \infty$, then:

$$\lim (a_n) = L$$

Assuming the sequence converges, we expect the terms to satisfy:

$$L = 1/2 (L + L) \Rightarrow L = L$$

which is consistent, but does not specify L. To find L explicitly, consider the homogeneous recurrence:

$$a_{n+1} = 1/2 (a_n + a_{n-1})$$

Rewrite as:

$$a_{n+1} - (1/2)\, a_n = (1/2)\, a_{n-1}$$

This suggests looking for solutions of the form:

$$a_n = r^n$$

Substituting into the recurrence:

$r^{n+1} = 1/2 (r^n + r^{n-1})$

Dividing both sides by r^{n-1}:

$r^2 = 1/2 (r + 1)$

which simplifies to:

$2 r^2 = r + 1$

or:

$2 r^2 - r - 1 = 0$

This quadratic equation yields:

$r = [1 \pm \sqrt{(1 + 8)}] / 4 = [1 \pm 3] / 4$

Calculating roots:

- $r = (1 + 3)/4 = 4/4 = 1$
- $r = (1 - 3)/4 = -2/4 = -1/2$

Thus, the general solution to the recurrence is:

$a_n = A (1)^n + B (-1/2)^n$

or

$a_n = A + B (-1/2)^n$

where A and B are constants determined by initial conditions.

Determination of Constants

Using initial conditions:

- For $n=1$: $a_1 = 1$

$a_1 = A + B (-1/2)^1 = A - (1/2) B = 1$

- For $n=2$: $a_2 = 2$

$a_2 = A + B (-1/2)^2 = A + (1/4) B = 2$

From these two equations:

1) $A - (1/2) B = 1$

2) $A + (1/4) B = 2$

Subtract equation 1 from equation 2:

$[A + (1/4) B] - [A - (1/2) B] = 2 - 1$

$(1/4) B + (1/2) B = 1$

$(1/4 + 1/2) B = 1$

$(3/4)B = 1$

$B = 4/3$

Substitute B back into equation 1:

$A - (1/2)(4/3) = 1$

$A - (2/3) = 1$

$A = 1 + (2/3) = (3/3) + (2/3) = 5/3$

Final Explicit Formula

The explicit formula for the sequence is:

$a_n = (5/3) + (4/3)(-1/2)^n$

Limit Behavior and Convergence

Limit as $n \to \infty$

Since $|-1/2| = 1/2 < 1$, the term $(-1/2)^n$ tends to zero as n approaches infinity. Therefore:

$\lim_{n \to \infty} a_n = 5/3 + (4/3)0 = 5/3$

The sequence converges to 5/3, approximately 1.6667.

Implications of the Result

- The explicit formula confirms that the sequence is bounded and converges.
- The initial oscillations caused by the $(-1/2)^n$ term diminish exponentially.
- The limit is independent of the initial variation, provided the initial values are finite.

Deeper Insights and Theoretical Significance

Stability and Fixed Point Analysis

The fixed point of the recurrence corresponds to the limit:

$L = 1/2 (L + L) \Rightarrow L = L$

which is trivial unless further conditions are imposed. The explicit solution demonstrates that the sequence approaches 5/3, aligning with the solution of the recurrence relation.

The stability of this fixed point can be analyzed through the characteristic roots:

- Since the dominant root is 1, the sequence's convergence depends on the other roots' magnitudes.
- The root at $-1/2$, with magnitude less than 1, ensures the exponential decay of oscillations.

Connection to Averaging and Smoothing

The recursive relation:

$a_{n+1} = 1/2 (a_n + a_{n-1})$

acts as a smoothing operator, akin to a simple moving average with lag. The sequence's convergence to a specific value reflects the averaging process's stabilizing effect.

Broader Context and Applications

Numerical Methods and Signal Processing

Sequences with recursive averaging are common in numerical algorithms, such as:

- Smoothing filters
- Moving averages
- Iterative solvers for differential equations

Understanding their convergence properties ensures the stability and accuracy of these methods.

Modeling in Economics and Natural Sciences

Such recursive relations can model:

- Population dynamics with generations averaging
- Economic indicators influenced by past values
- Physical systems with damping or equilibrium tendencies

The explicit formula and convergence analysis provide a foundation for such applications.

Summary of Main Findings

- The sequence defined by A1=1, A2=2, and An+1=1/2 (an + an-1) converges.
- It explicitly converges to 5/3 (approximately 1.6667).
- The explicit formula: an = (5/3) + (4/3) (-1/2)^n.
- The convergence is exponentially fast, driven by the root at −1/2.

Conclusion

The detailed investigation into the sequence (an) reveals a beautiful interplay between recursive definitions, explicit solutions, and convergence behavior. The sequence's explicit formula not only confirms its convergence but also provides insight into the rate of convergence and oscillatory nature. This analysis exemplifies how recursive relations, even simple ones, harbor rich mathematical

[Consider The Sequence An Given By A1 1 A2 2 An 1 1 2 An An 1 For N Gt 2 We Will Show That]

Consider The Sequence An Given By A1 1 A2 2 An 1 1 2 An An 1 For N Gt 2 We Will Show That

Related Articles

- [Cuicuilco And Teotihuacan Were Rival City-states In The Valley Of Mexico In Conflict With One]

Another
- Find (fg)(4), Given The Following Functions: F(x)=2x+1 G(x)=x ^2+3x A) 57 B) 55 C) 20 D) 45 E) 108 F)
- Consider The Following Consumers Problem: U(x,y) = X1/4 Y3/4. Prices Are Px =$10, Py = $10 And Income

Back to Home