Consider The Solution Of The Differential Equationy=3ypassing Throughy(0)=0.5. Sketch The Slope Field

Consider The Solution Of The Differential Equationy=3ypassing Throughy(0)=0.5. Sketch The Slope Field is an intriguing problem that combines the analytical solving of differential equations with the graphical intuition provided by slope fields. This type of problem is fundamental in understanding how solutions behave without necessarily finding explicit formulas, especially for first-order differential equations. In this article, we will explore the solution process step-by-step, interpret the behavior of the solution through the slope field, and discuss methods for sketching the slope field accurately.

Understanding the Differential Equation

Formulation of the Given Differential Equation

The differential equation in question is:

$$
\frac{dy}{dx} = 3y
$$

This is a first-order linear differential equation, which is separable and can be solved analytically. The initial condition provided is:

$$
y(0) = 0.5
$$

The goal is to find the particular solution that satisfies this initial condition and then visualize the behavior of solutions through the slope field.

Significance of the Equation

This differential equation models exponential growth or decay depending on the context. Since the coefficient is positive (3), the solutions tend to grow exponentially as x increases. The initial condition sets the starting point at $x=0$, $y=0.5$.

Solving the Differential Equation

Method: Separation of Variables

The differential equation $\frac{dy}{dx} = 3y$ can be written as:

$$
\frac{dy}{y} = 3\,dx
$$

Integrating both sides:

$$
\int \frac{1}{y}\,dy = \int 3\,dx
$$

$$
\ln|y| = 3x + C
$$

Exponentiating both sides:

$$
|y| = e^{3x + C} = e^{C} e^{3x}
$$

Let $A = e^{C}$, which is a positive constant. Therefore, the general solution is:

$$
y = A e^{3x}
$$

Applying the initial condition $y(0) = 0.5$:

$$
0.5 = A e^{0} \Rightarrow A = 0.5
$$

Thus, the particular solution is:

$$
\boxed{ y(x) = 0.5 e^{3x} }
$$

This explicit solution describes how y evolves with respect to x.

Interpreting the Solution and Behavior

Growth Pattern

Since the solution involves an exponential with a positive exponent, it indicates rapid growth of y as x increases. Conversely, as x approaches negative infinity, the solution approaches zero, reflecting exponential decay backwards in x.

Behavior at the Initial Point

At $x=0$, $y=0.5$, matching the initial condition. The solution's slope at this point is:

$$
\frac{dy}{dx} = 3 \times 0.5 = 1.5
$$

which indicates a positive slope, meaning the solution curve is increasing at $x=0$.

Sketching the Slope Field

The slope field (also called a direction field) provides a visual representation of the solutions to the differential equation without solving it explicitly.

What Is a Slope Field?

A slope field consists of small line segments or arrows at grid points (x, y), where each segment's slope corresponds to the value of $\frac{dy}{dx}$ at that point. It visually indicates the direction in which solutions pass through the point.

Steps to Sketch the Slope Field for $dy/dx=3y$

1. Select a grid of points: Choose a range of x and y values, such as $-1 \leq x \leq 2$ and $0 \leq y \leq 2$.

2. Calculate slopes at each point: For each (x, y), compute:

$$
\text{Slope} = 3 y
$$

3. Draw small line segments: At each point, draw a tiny line with the calculated slope.

4. Observe the pattern: The line segments will show that the slopes increase with y and are zero when $y=0$.

Expected Pattern of the Slope Field

- When $y=0$, the slope is zero; the line segments are horizontal.
- For positive y, the slopes are positive and increase as y increases.
- For small y, slopes are gentle; for large y, slopes are steep, indicating rapid growth.

Sketching the Solution Curve through the Slope Field

Starting with the Initial Condition

Plot the initial point $(0, 0.5)$. Here, the slope of the solution is 1.5, so the solution curve should pass through this point with a moderate upward tilt.

Following the Direction of the Slope Field

- From the initial point, draw a curve that follows the direction of the small line segments.
- The curve should steepen rapidly as x increases because y grows exponentially.
- For negative x, the solution approaches zero, reflecting the exponential decay.

Visualizing the Solution

The solution curve will resemble an exponential growth curve starting at $(0, 0.5)$ and increasing rapidly as x increases, aligning with the explicit solution $y(x) = 0.5 e^{3x}$.

Applications and Significance of Understanding the Solution and Slope Field

Real-world Applications

Differential equations like $dy/dx = 3 y$ appear in models involving exponential growth or decay, such as population dynamics, radioactive decay, and compound interest calculations.

Importance of Slope Fields in Teaching and Analysis

- Visual tools like slope fields help students intuitively grasp the behavior of solutions.
- They allow for qualitative analysis without solving the differential equation explicitly.
- Slope fields are particularly useful when explicit solutions are difficult or impossible to find.

Conclusion

In summary, the differential equation $dy/dx = 3 y$ with initial condition $y(0) = 0.5$ yields an exponential solution $y(x) = 0.5 e^{3x}$. Its behavior demonstrates exponential growth, with the slope at any point proportional to the current value of y. Sketching the slope field reinforces understanding by visually depicting how solutions evolve, starting from the initial point and following the directions indicated by the small line segments. Mastering both analytical solutions and graphical interpretations like slope fields equips learners with a comprehensive understanding of differential equations and their applications in various scientific fields.

Frequently Asked Questions

What is the differential equation given in the problem?

The differential equation is $y' = 3y$.

What initial condition is provided for the differential equation?

The initial condition is $y(0) = 0.5$.

What is the general solution to the differential equation y' = 3y?

The general solution is $y(t) = Ce^{3t}$, where C is a constant determined by initial conditions.

How do we find the particular solution passing through y(0) = 0.5?

Substitute $t=0$ and $y(0)=0.5$ into the general solution: $0.5 = C e^{0} = C$, so $C=0.5$. The particular solution is $y(t)= 0.5 e^{3t}$.

What does the slope field for y' = 3y look like?

The slope field consists of small line segments with slopes proportional to y at each point, increasing exponentially as y increases.

How can sketching the slope field help understand the solution y(t) = 0.5 e^{3t}?

The slope field visually illustrates how solutions grow exponentially from y=0.5 at t=0, confirming the behavior of the particular solution.

What is the significance of the initial condition y(0)=0.5 in the solution?

It determines the specific exponential curve among the family of solutions, fixing the constant C=0.5.

How does the solution y(t) = 0.5 e^{3t} behave as t approaches infinity?

The solution grows exponentially without bound as t increases, due to the positive exponential term.

Is the solution y(t) = 0.5 e^{3t} stable or unstable?

The solution is unstable; small increases in initial value can lead to rapid growth, characteristic of exponential solutions with positive growth rates.

What are some practical applications of solving differential equations like y' = 3y?

Such equations model exponential growth or decay processes, including population dynamics, radioactive decay, and compound interest calculations.

Additional Resources

Consider The Solution Of The Differential Equation y = 3y passing Through y(0) = 0.5. Sketch The Slope Field

Understanding differential equations and their solutions is fundamental in many fields, including physics, engineering, and biology. The specific differential equation y = 3y, along with the initial condition y(0) = 0.5, offers an insightful example to explore the behavior of solutions, the nature of slope fields, and the process of sketching solutions graphically. This article provides a comprehensive analysis of this differential equation, its solution, and how to visualize it through slope fields, along with a detailed discussion of the concepts involved, their applications, and their limitations.

Introduction to the Differential Equation y = 3y

The differential equation y = 3y is a simple, first-order linear differential equation that exemplifies exponential growth or decay depending on the context. The equation states that the rate of change of y with respect to x (or the independent variable) is proportional to y itself, with a constant proportionality factor of 3.

Key features:
- Type of Equation: First-order linear differential equation
- Form: dy/dx = 3y
- Interpretation: The rate of change of y is directly proportional to y

Implications:
- Solutions tend to be exponential functions
- The proportionality constant (3) determines the growth rate

This form is characteristic of many natural phenomena, such as population growth, radioactive decay, or compound interest, where the rate of change depends on the current value.

Solving the Differential Equation y = 3y

To find the explicit solution, we treat the differential equation:

dy/dx = 3y

This is a separable differential equation, which allows us to separate the variables y and x:

dy/y = 3 dx

Step-by-step solution:

1. Separate variables:

\[
\frac{dy}{y} = 3\, dx
\]

2. Integrate both sides:

\[
\int \frac{1}{y} dy = \int 3\, dx
\]

\[
\ln |y| = 3x + C
\]

3. Solve for y:

\[

y = e^{3x + C} = e^{C} e^{3x}
\]

Let \(A = e^{C}\), which is an arbitrary constant:

\[
y = A e^{3x}
\]

Applying the initial condition y(0) = 0.5:

\[
0.5 = A e^{0} = A
\]

Thus, A = 0.5, and the particular solution is:

\[
\boxed{
y(x) = 0.5 e^{3x}
}
\]

Analysis of the solution:
- This exponential function models rapid growth as x increases.
- At x = 0, y = 0.5, matching the initial condition.
- As x decreases, y approaches zero, indicating decay toward zero in negative x directions.

Features and Behavior of the Solution

The explicit solution, \(y(x) = 0.5 e^{3x} \), exhibits several key features:

Exponential Growth

Since the exponent is positive (3), the solution grows exponentially as x increases. This signifies a system where the rate of change accelerates proportionally to the current state.

Initial Condition

The initial value y(0) = 0.5 anchors the solution at x = 0, providing a specific trajectory among infinitely many solutions.

Asymptotic Behavior

- As \(x \to +\infty \), \(y(x) \to +\infty \), indicating unbounded growth.
- As \(x \to -\infty \), \(y(x) \to 0 \), approaching zero but never crossing it, showing decay toward zero.

Stability

The solution is unstable in the sense that small variations in initial conditions can lead to significantly different trajectories over large x values, typical of exponential solutions.

Slope Field: Concept and Construction

A slope field (or direction field) offers a powerful visual tool to understand differential equations without solving them explicitly. For the given differential equation, the slope field illustrates the slopes of the solution curves at various points.

What is a slope field?

- A grid of points in the xy-plane
- At each point, a small line segment is drawn with a slope equal to dy/dx at that point
- The collection of these tiny line segments visually suggests the shape of the solution curves passing through the grid

How to sketch the slope field for y = 3y

Since the differential equation is dy/dx = 3y, the slope at any point (x, y) depends solely on y:

$$
\text{Slope} = 3 y
$$

This simplifies the construction:

- At points where y = 0, the slope is zero, so horizontal line segments are drawn.
- At points where y > 0, the slope is positive and proportional to y, increasing as y increases.
- At points where y < 0 (though outside the initial condition), the slope is negative, indicating decreasing y.

Key observations:
- The slope field is symmetric about the x-axis because the slope depends linearly on y.
- The slope at any point depends only on y, not on x, indicating that solution curves are exponential functions shifted vertically.

Sketching the slope field:

1. Select grid points: Choose a set of points covering the region of interest.
2. Calculate slopes: For each point, compute the slope $3 y$.
3. Draw small line segments: At each point, draw a tiny line segment with the computed slope.
4. Identify solution curves: Observe how these segments align to form exponential growth curves passing through the initial point.

Visual features:

- Near y = 0, the slopes are near zero, so the solution curves are nearly horizontal.
- As y increases, the slopes steepen, indicating rapid exponential growth.
- The solution passing through y(0) = 0.5 will start at (0, 0.5) and curve upward steeply as x increases.

Interpreting the Solution and Slope Field

The explicit solution and the slope field collectively provide a comprehensive understanding of the differential equation's behavior.

How the slope field confirms the solution:

- The solution $y(x) = 0.5 e^{3x}$ is a smooth, increasing exponential curve passing through (0, 0.5).
- The tangent lines in the slope field at points along this curve align with the slope given by the solution's derivative $dy/dx = 3 y$.
- The slope field visually emphasizes the exponential growth, with steeper slopes at higher y-values.

Advantages of visualizing with slope fields:

- Understand qualitative behavior without solving explicitly.
- Explore solutions for different initial conditions.
- Identify equilibrium solutions, such as y = 0, which corresponds to a flat line with zero slope.

Limitations:

- Slope fields are only qualitative; precise solutions require algebraic methods.
- For complex differential equations, slope fields can become cluttered or difficult to interpret.
- The accuracy depends on the density of the grid and clarity of the sketch.

Applications and Broader Context

The differential equation y = 3y and its solution have broad applications:

- Population Dynamics: Exponential growth when resources are unlimited.
- Radioactive Decay & Nuclear Processes: When modeled with negative growth rates.
- Finance: Compound interest with continuous growth.

Educational significance:

- Demonstrates the fundamental concept of exponential functions.
- Highlights the importance of initial conditions in differential equations.

- Serves as a foundational example for more complex systems.

Practical considerations:

- In real-world scenarios, exponential growth often cannot continue indefinitely due to resource constraints.
- Modifying the differential equation to include additional factors (e.g., logistic growth) offers more realistic models.

Conclusion

The differential equation y = 3y, with initial condition y(0) = 0.5, exemplifies exponential growth behavior with a clear, explicit solution: $y(x) = 0.5 e^{3x}$. Visualizing this solution through a slope field provides intuitive insight into its dynamics, illustrating how the solution curves evolve and confirming the exponential nature predicted by the algebraic solution. The process of constructing and interpreting slope fields enhances understanding of differential equations beyond mere formulas, fostering a deeper appreciation for their applications in modeling natural phenomena.

While the explicit solution offers precise quantitative insight, the slope field emphasizes qualitative behavior, stability, and the influence of initial conditions. Together, these tools serve as essential components in the mathematician's toolkit for analyzing differential equations. As with all models, it is crucial to recognize their limitations and the contexts where they are applicable, ensuring accurate and meaningful interpretations in scientific and engineering contexts.

Features Summary:

- Explicit Solution: $y(x) = 0.5 e^{3x}$
- Growth Type: Exponential
- Initial Condition: y(0) = 0.5
- Slope Field: Visual representation of the differential equation's behavior
- Applications: Population growth, finance, physics
- Limitations:

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