Determine The Radius And Interval Of Convergence Of The Following Series... SERIES ANSWERS) . (x–1)"

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Understanding the convergence properties of power series is fundamental in analysis, particularly when studying functions represented by infinite series. Power series centered at a specific point, such as (x - 1), are invaluable tools for approximating functions within certain regions of the real line or the complex plane. Determining the radius and interval of convergence provides insight into where these series are valid (i.e., converge to the function) and where they diverge. In this comprehensive guide, we explore methods to find the radius and interval of convergence for series centered at (x - 1), supported by detailed examples and explanations.

Fundamentals of Power Series and Convergence

What Is a Power Series?

A power series centered at a point a is an infinite sum of the form:

$$
\sum_{n=0}^{\infty} c_n (x - a)^n
$$

where:
- c_n are coefficients,
- x is a variable,
- a is the center of the series (here, $a = 1$).

Power series can represent many functions within their radius of convergence, making them a cornerstone of analysis.

Importance of Radius and Interval of Convergence

- Radius of Convergence R: The distance from the center a within which the series converges absolutely.
- Interval of Convergence: The set of x values for which the series converges, which includes the points where the series converges conditionally or absolutely.

Knowing R and the interval helps determine the applicability of the series to approximate functions and analyze their behavior.

Methods to Determine the Radius of Convergence

The primary tools for calculating the radius of convergence are:

1. The Root Test (Cauchy-Hadamard Theorem)

The Cauchy-Hadamard formula states:

$$
\frac{1}{R} = \limsup_{n \to \infty} |c_n|^{1/n}
$$

where:
- R is the radius of convergence,
- c_n are the coefficients of the series.

Procedure:
1. Find the general form of c_n.
2. Calculate $\limsup_{n \to \infty} |c_n|^{1/n}$.
3. Take the reciprocal to find R.

Example:
Given $c_n = \frac{1}{n!}$, then:

$$
|c_n|^{1/n} = \left(\frac{1}{n!} \right)^{1/n}
$$

Since $n!$ grows faster than any exponential, $|c_n|^{1/n} \to 0$, so:

$$
\frac{1}{R} = 0 \Rightarrow R = \infty
$$

This series converges for all real x.

2. The Ratio Test

The Ratio Test involves:

$$
L = \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right|
$$

- If L exists, then the radius of convergence is:

$$
R = \frac{1}{L}
$$

- For the series $\sum c_n (x - a)^n$, the convergence depends on $|x - a| < R$.

Procedure:
1. Compute $\left| \frac{c_{n+1}}{c_n} \right|$.
2. Take the limit as $n \to \infty$.
3. Find $R = \frac{1}{L}$.

Example:
For $c_n = \frac{1}{2^n}$, then:

$$
\left| \frac{c_{n+1}}{c_n} \right| = \frac{1/2^{n+1}}{1/2^n} = \frac{1}{2}
$$

which is constant, so $L = 1/2$, and:

$$
R = \frac{1}{L} = 2
$$

Determining the Interval of Convergence

Once the radius R is found, the next step is to identify the actual interval where the series converges, including endpoints.

Testing Endpoints

Because the radius test only indicates convergence within $(a - R, a + R)$, endpoints $x = a \pm R$ require separate analysis:

- Substitute $x = a \pm R$ into the series.
- Use appropriate convergence tests (e.g., p-test, alternating series test, comparison test) to determine if the series converges at these points.

Step-by-Step Example: Series Centered at (x - 1)

Let's consider an example series:

$$
\sum_{n=0}^{\infty} \frac{(x - 1)^n}{n!}
$$

Step 1: Identify Coefficients c_n

Here,

$$
c_n = \frac{1}{n!}
$$

and the series is centered at $a=1$.

Step 2: Find the Radius of Convergence R Using the Root Test

Calculate:

$$
|c_n|^{1/n} = \left(\frac{1}{n!} \right)^{1/n}
$$

As $n \to \infty$, $n!$ grows faster than any exponential function, so:

$$
|c_n|^{1/n} \to 0
$$

Thus,

$$
\frac{1}{R} = 0 \Rightarrow R = \infty
$$

Conclusion: The series converges for all $x \in \mathbb{R}$.

Step 3: Determine the Interval of Convergence

Since $R = \infty$, the series converges everywhere, and the interval is:

$$
(-\infty, \infty)
$$

Another Example: Series with a Finite Radius

Consider:

$$
\sum_{n=0}^{\infty} n (x - 1)^n
$$

Step 1: Coefficients $c_n = n$

Step 2: Find R Using the Ratio Test

Compute:

$$
\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{n+1}{n} \right| \to 1 \quad \text{as } n \to \infty
$$

The limit $L = 1$, so:

$$
R = \frac{1}{L} = 1
$$

Step 3: Determine the Interval of Convergence

The series converges for:

$$
|x - 1| < 1
$$

or

$$
0 < x < 2
$$

Now, test endpoints:

- At $x=0$:

$$
\sum_{n=1}^{\infty} n (0 - 1)^n = \sum_{n=1}^{\infty} n (-1)^n
$$

This is an alternating series with terms n, which do not tend to zero. Since the terms do not tend to zero, the series diverges at $x=0$.

- At $x=2$:

$$
\sum_{n=1}^{\infty} n (2 - 1)^n = \sum_{n=1}^{\infty} n (1)^n = \sum_{n=1}^{\infty} n
$$

\]

which diverges.

Final conclusion:

- The interval of convergence is open:

$$
(1 - 1, 1 + 1) = (0, 2)
$$
- Endpoints are excluded since the series diverges at both.

Summary of the Procedure

To systematically determine the radius and interval of convergence for a series centered at $(x - 1)$:

1. Identify the coefficients c_n.
2. Apply the Root Test or Ratio Test to find R:
    - Root Test: $R = 1 / \limsup_{n \to \infty} |c_n|^{1/n}$
    - Ratio Test: $R = 1 / \lim_{n \to \infty} |c_{n+1}/c_n|$
3. State the interval as $(a - R, a + R)$.
4. Test the endpoints $x = a \pm R$ separately to determine whether they are included.