Determine Whether The Geometric Series Is Convergent Or Divergent. If It Is Convergent, Find The Sum.

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Understanding the behavior of geometric series is fundamental in mathematics, especially in calculus and algebra. These series appear frequently in various applications, including finance, physics, computer science, and engineering. The key question often posed is whether a given geometric series converges (approaches a finite sum) or diverges (grows without bound). Furthermore, if the series converges, determining its sum is essential for practical calculations. This comprehensive guide aims to explain the criteria for convergence, how to identify whether a geometric series converges or diverges, and the method to compute its sum when applicable.

What Is a Geometric Series?

A geometric series is an infinite sum of terms where each term is obtained by multiplying the previous term by a fixed, non-zero constant called the common ratio. Formally, a geometric series can be expressed as:

$$ S = a + ar + ar^2 + ar^3 + \dots $$

where:
- a is the first term of the series,
- r is the common ratio between successive terms,
- n indicates the term number (starting from 0).

This series can be finite or infinite, but our focus is primarily on infinite series, as their convergence properties are more nuanced and mathematically interesting.

Understanding Convergence and Divergence

The main concern when analyzing a geometric series is whether the sum approaches a finite limit as the number of terms tends to infinity.

Convergence occurs when the sum approaches a specific finite value, meaning the series "settles" to a limit.

Divergence occurs when the sum grows without bound or oscillates, never approaching a finite limit.

Criteria for Convergence of a Geometric Series

The convergence of an infinite geometric series hinges on the value of the common ratio r.

The Convergence Criterion:

An infinite geometric series converges if and only if:

$$|r| < 1$$

Conversely, if:

$$|r| \geq 1$$

then the series diverges.

Why Does This Criterion Hold?

- When $|r| < 1$, the terms ar^n tend to zero as $n \to \infty$, which allows the series to sum to a finite value.
- When $|r| \geq 1$, the terms do not tend to zero (they stay constant or grow), preventing the series from converging.

How to Determine Whether a Geometric Series Converges or Diverges

Given the first term a and the common ratio r, follow these steps:

1. Identify a and r:
Extract the first term and ratio from the series.

2. Calculate $|r|$:
Find the absolute value of the common ratio.

3. Apply the convergence criterion:
- If $|r| < 1$, the series converges.
- If $|r| \geq 1$, the series diverges.

4. Optional - Check the partial sums:
For confirmation, compute the partial sums for increasing n to observe behavior.

Calculating the Sum of a Convergent Geometric Series

When the series converges ($|r| < 1$), the sum of the infinite series can be derived using the formula:

$$S = \frac{a}{1 - r}$$

This formula is valid under the convergence condition and provides a quick way to evaluate the sum.

Derivation of the Sum Formula:

The sum of the first n terms of a geometric series is:

$$S_n = a \frac{1 - r^n}{1 - r}$$

As $n \to \infty$, if $|r| < 1$, then $r^n \to 0$, leading to:

$$S = \lim_{n \to \infty} S_n = \frac{a}{1 - r}$$

Examples Demonstrating Convergence and Divergence

Example 1: Convergent Series

Suppose the series:

$$3 + 1.5 + 0.75 + 0.375 + \dots$$

Identify a and r:

- $a = 3$
- To find r:

$$r = \frac{1.5}{3} = 0.5$$

Since $|r| = 0.5 < 1$, the series converges.

Calculate the sum:

$$S = \frac{a}{1 - r} = \frac{3}{1 - 0.5} = \frac{3}{0.5} = 6$$

Conclusion: The series converges to 6.

Example 2: Divergent Series

Consider:

$$5 + 5 + 5 + \dots$$

Here:

- $a = 5$
- $r = 1$

Since $|r| = 1$, the series diverges; the sum grows infinitely large as more terms are added.

Special Cases and Important Notes

- When $r = -1$: The series oscillates between two values, and the sum does not tend to a finite limit; it diverges.
- When $r = 1$: The series is simply $a + a + a + \dots$, which diverges unless $a = 0$.
- Finite series: If the series has a finite number of terms, the sum is straightforward to compute directly.

Applications of Geometric Series Analysis

Understanding whether a geometric series converges has practical implications:

- Financial calculations: Present value of perpetuities.
- Physics: Decay processes.
- Computer science: Algorithm analysis, such as analyzing recursive algorithms.
- Engineering: Signal processing and control systems.

Summary and Key Takeaways

- The convergence of an infinite geometric series depends solely on the magnitude of the common ratio r.
- If $|r| < 1$, the series converges, and the sum is $S = \frac{a}{1 - r}$.
- If $|r| \geq 1$, the series diverges.
- The initial term a and ratio r are essential to determine the series' behavior.

Conclusion

Determining whether a geometric series converges or diverges is a fundamental skill in mathematics. By analyzing the common ratio r, one can quickly assess the series' behavior. When the series converges, calculating its sum using the formula $\frac{a}{1 - r}$ provides essential insights into the series' total value. Mastery of these concepts is vital for solving complex problems in various scientific and engineering fields. Always verify the conditions for convergence before attempting to compute the sum to ensure accurate results.

Keywords: geometric series, convergence, divergence, sum of geometric series, common ratio, infinite series, mathematical series, series analysis, calculus, algebra.

Frequently Asked Questions

How do you determine if a geometric series converges or diverges?

A geometric series converges if the absolute value of the common ratio $|r|$ is less than 1; otherwise, it diverges.

What is the formula to find the sum of a convergent geometric series?

The sum S of a convergent geometric series with first term a and common ratio r ($|r|<1$) is $S = a / (1 - r)$.

Given the series 3 + 1.5 + 0.75 + ..., is it convergent? If so, what is its sum?

Yes, it converges because $|r| = 0.5 < 1$. The sum is $S = 3 / (1 - 0.5) = 3 / 0.5 = 6$.

For the series 5 + 10 + 20 + ..., does it converge or diverge?

It diverges because the common ratio $r = 2$ has $|r| > 1$, so the series does not have a finite sum.

How can you determine the sum if the common ratio r is negative, for example r = -0.8?

If $|r| < 1$, the geometric series converges, and the sum is $S = a / (1 - r)$. The sign of r affects the series' terms but not the convergence criterion.

What is the significance of the common ratio's magnitude in geometric series convergence?

The magnitude $|r|$ determines convergence: if $|r| < 1$, the series converges; if $|r| \geq 1$, it diverges.

Can a geometric series with r = 1 converge?

No, because if $r = 1$, the terms do not approach zero, and the series diverges.

If the first term a = 2 and the common ratio r = 0.3, what is the sum of the series?

Since $|r| < 1$, the series converges. The sum is $S = 2 / (1 - 0.3) = 2 / 0.7 \approx 2.857$.

Additional Resources

Determine Whether The Geometric Series Is Convergent Or Divergent. If It Is Convergent, Find The Sum.

Understanding the behavior of infinite series is a cornerstone of advanced mathematics, especially in fields like calculus, engineering, and finance. Among these series, geometric series hold particular significance due to their simple structure and wide range of applications. The question that often arises is: Does a given geometric series converge or diverge? If it converges, what is its sum? This article aims to demystify these questions, providing a clear, technical, yet accessible explanation of how to analyze geometric series for convergence and, when appropriate, compute their sums.

What Is a Geometric Series?

Before delving into convergence criteria, it’s essential to understand what a geometric series is.

Definition:
A geometric series is an infinite sum of the form:

$$ S = a + ar^2 + ar^3 + \dots $$

where:
- a is the first term of the series,
- r is the common ratio between successive terms.

Each term after the first is obtained by multiplying the previous term by r. The series extends infinitely, and the key question is whether this sum approaches a finite value (converges) or grows without bound (diverges).

Finite vs. Infinite Series:

- Finite geometric series involve a finite number of terms and have a straightforward sum formula.
- Infinite geometric series involve infinitely many terms, and their sum depends on the value of r.

The Convergence Criterion for Geometric Series

The behavior of an infinite geometric series hinges critically on the common ratio r. The fundamental question is: Under what condition does the infinite sum have a finite value?

Key Theorem:
An infinite geometric series converges if and only if:

$$ |r| < 1 $$

Otherwise, it diverges.

Intuition:
- When $|r| < 1$, the terms ar^n tend toward zero as $n \to \infty$, allowing the series to settle on a finite sum.
- When $|r| \geq 1$, the terms do not tend toward zero; instead, they either stay the same (if $r = 1$) or grow without bound (if $|r| > 1$), causing the sum to diverge.

Mathematical Derivation of Convergence and Sum

1. Sum of a Finite Geometric Series
For a finite number of terms n, the sum S_n is:

$$ S_n = a \frac{1 - r^n}{1 - r} \quad \text{(for } r \neq 1 \text{)} $$

As $n \to \infty$:

- If $|r| < 1$, then $r^n \to 0$, so:

$$ \lim_{n \to \infty} S_n = \frac{a}{1 - r} $$

- If $|r| \geq 1$, the limit does not exist or is infinite, indicating divergence.

2. Sum of an Infinite Geometric Series
Therefore, the sum to infinity (when it exists) is:

$$ S_{\infty} = \frac{a}{1 - r} \quad \text{for } |r| < 1 $$

This formula is powerful because it provides a closed-form expression for the sum of an infinite series, which is often useful in calculations across various disciplines.

Practical Steps to Determine Convergence and Calculate the Sum

To analyze whether a given geometric series converges and to find its sum if it does, follow these steps:

Step 1: Identify a and r
- Extract the first term a.
- Determine the common ratio r by dividing any term by its previous term.

Step 2: Check the magnitude of r
- If $|r| < 1$, the series converges.
- If $|r| \geq 1$, the series diverges.

Step 3: Calculate the sum if convergent
- Use $S_{\infty} = \frac{a}{1 - r}$.

Examples and Applications

Example 1: Convergent Series

Suppose the series is:

$$3 + 1.5 + 0.75 + 0.375 + \dots$$

- $a = 3$
- $r = \frac{1.5}{3} = 0.5$

Since $|0.5| < 1$, the series converges.

Calculate the sum:

$$S_{\infty} = \frac{3}{1 - 0.5} = \frac{3}{0.5} = 6$$

Thus, the sum of the series is 6.

Example 2: Divergent Series

Consider:

$$2 + 2 + 2 + \dots$$

- $a = 2$
- $r = 1$

Since $|r| = 1$, the series diverges; the sum does not approach a finite value.

Application in Finance:

Geometric series are fundamental in calculating the present value of perpetuities or the total value of a series of payments that grow or decline geometrically over time.

Application in Physics:
Modeling decay processes or population dynamics often involves geometric series, where the convergence determines whether the total quantity stabilizes or diverges over time.

Common Pitfalls and Caveats

- Misidentifying r: Always verify the common ratio by dividing successive terms; errors here lead to incorrect conclusions.

- Ignoring the magnitude of r: The convergence depends on $|r|$, not just r.

- Zero or negative ratios: The convergence criteria still hold. For negative ratios, the series may oscillate, but if $|r| < 1$, the series converges.

- Special cases: When $r = 1$, the series is simply:

$$ a + a + a + \dots $$

which diverges unless $a = 0$.

Summary and Final Thoughts

Determining whether a geometric series converges or diverges is a fundamental skill in mathematical analysis. The core criterion is the magnitude of the common ratio r:

- Converges if $|r| < 1$
- Diverges if $|r| \geq 1$

When it converges, the sum to infinity can be succinctly calculated using:

$$ \boxed{S_{\infty} = \frac{a}{1 - r}} $$

This formula offers a powerful tool for applications across various scientific and engineering domains, enabling practitioners to analyze infinite processes, model real-world phenomena, and solve complex problems efficiently.

Understanding the behavior of geometric series not only deepens mathematical insight but also provides practical means to handle infinite sequences in theoretical and applied contexts. Whether in calculating the present value of financial instruments, analyzing decay rates, or solving recurrence relations, the principles outlined here serve as essential building blocks for advanced mathematical reasoning.

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