Evaluate The Line Integral, Where C Is The Given Curve. C X Sin(y) Ds, C Is The Line Segment From (0,

Evaluate The Line Integral, Where C Is The Given Curve. C X Sin(y) Ds, C Is The Line Segment From (0,

Introduction to Line Integrals and Their Significance

Line integrals are a fundamental concept in multivariable calculus, playing a critical role in fields such as physics, engineering, and mathematics. They allow us to compute the accumulation of a function along a curve in a plane or space, providing insights into work done by a force field, mass distribution, or other physical phenomena.

In this article, we will explore the evaluation of a specific line integral where the integrand involves a product of the x-coordinate and the sine of y, integrated along a line segment from a starting point to an endpoint. We will delve into the mathematical process, step-by-step, to provide a comprehensive understanding of how to evaluate such integrals.

Understanding the Problem: The Given Line Integral

The line integral under consideration is of the form:

$$
\int_C x \sin(y) \, ds
$$

where:

- C is a line segment from $(0, y_0)$ to (x_1, y_1),
- ds is the differential element of arc length along C,
- x and y are the coordinates of points on the curve C.

The goal is to evaluate this integral explicitly, given specific endpoints for the line segment C.

Note: For clarity, the initial point is $(0, y_0)$. The endpoint can be specified as (x_1, y_1), with both x_1 and y_1 being known constants.

Step 1: Parameterizing the Curve C

To evaluate a line integral, the first step is to parametrize the curve C. Since C is a straight line segment from $(0, y_0)$ to (x_1, y_1), a linear parametrization is appropriate.

Let the parameter be t, where:

$$
t \in [0, 1]
$$

The parametric equations for C are:

$$
x(t) = 0 + t (x_1 - 0) = t x_1
$$
$$
y(t) = y_0 + t (y_1 - y_0)
$$

This parametrization maps $t = 0$ to the initial point $(0, y_0)$, and $t = 1$ to the endpoint (x_1, y_1).

Step 2: Computing ds in Terms of t

The differential arc length element ds on the curve is given by:

$$
ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
$$

Calculating the derivatives:

$$
\frac{dx}{dt} = x_1
$$
$$
\frac{dy}{dt} = y_1 - y_0
$$

Therefore,

$$
ds = \sqrt{x_1^2 + (y_1 - y_0)^2} \, dt
$$

This constant represents the length of the line segment scaled by the parameter t.

Step 3: Expressing the Integrand in Terms of t

Recall the integrand:

$$
x \sin(y)
$$

Substituting $x(t)$ and $y(t)$:

$$
x(t) = t x_1
$$
$$
y(t) = y_0 + t (y_1 - y_0)
$$

Thus, the integrand becomes:

$$
x(t) \sin(y(t)) = t x_1 \sin\left(y_0 + t (y_1 - y_0) \right)
$$

Step 4: Setting Up the Integral

Now, the line integral transforms into:

$$
\int_{t=0}^{1} x(t) \sin(y(t)) \, ds = \int_{0}^{1} t x_1 \sin\left(y_0 + t (y_1 - y_0) \right)
\sqrt{x_1^2 + (y_1 - y_0)^2} \, dt
$$

Factoring constants:

$$
\sqrt{x_1^2 + (y_1 - y_0)^2} \, x_1 \int_0^1 t \sin\left(y_0 + t (y_1 - y_0) \right) dt
$$

Define:

$$
L = \sqrt{x_1^2 + (y_1 - y_0)^2}
$$

so the integral simplifies to:

$$
L \, x_1 \int_0^1 t \sin\left(y_0 + t (y_1 - y_0) \right) dt
$$

\]

Our task reduces to evaluating:

\[
I = \int_0^1 t \sin(y_0 + t (y_1 - y_0)) dt
\]

Step 5: Computing the Integral \(I \)

Let \(A = y_0 \) and \(B = y_1 - y_0 \). Then,

\[
I = \int_0^1 t \sin(A + B t) dt
\]

This integral involves a product of \(t \) and a sine function with a linear argument. To evaluate \(I \), we can use integration by parts.

Integration by parts:

Let:

- \(u = t \Rightarrow du = dt \),
- \(dv = \sin(A + B t) dt \).

We need to find \(v \):

\[
v = \int \sin(A + B t) dt
\]

Substituting \(u = A + B t \):

\[
dv = \sin u \, dt
\]

\[
v = - \frac{1}{B} \cos u + C
\]

(We can ignore the constant of integration since we're evaluating a definite integral.)

Now, applying integration by parts:

\[
I = u v \Big|_0^1 - \int_0^1 v du
\]

\[
I = t \left(- \frac{1}{B} \cos(A + B t) \right) \Big|_0^1 + \frac{1}{B} \int_0^1 \cos(A + B t) dt
\]

Evaluating the first term:

\[
- \frac{t}{B} \cos(A + B t) \Big|_0^1 = - \frac{1}{B} \cos(A + B) + 0
\]

since at \(t=0 \):

\[
- \frac{0}{B} \cos(A) = 0
\]

The remaining integral:

\[
\frac{1}{B} \int_0^1 \cos(A + B t) dt
\]

which evaluates to:

\[
\frac{1}{B} \times \frac{1}{B} \left[\sin(A + B t) \right]_0^1 = \frac{1}{B^2} \left(\sin(A + B) - \sin A \right)
\]

Putting it all together:

\[
I = - \frac{1}{B} \cos(A + B) + \frac{1}{B^2} \left(\sin(A + B) - \sin A \right)
\]

Recall that:

\[
A = y_0, \quad B = y_1 - y_0
\]

Thus,

\[
I = - \frac{1}{y_1 - y_0} \cos(y_0 + (y_1 - y_0)) + \frac{1}{(y_1 - y_0)^2} \left(\sin(y_0 + (y_1 - y_0)) - \sin y_0 \right)
\]

Simplify:

\[

\cos(y_0 + y_1 - y_0) = \cos(y_1)
\]
\[
\sin(y_0 + y_1 - y_0) = \sin(y_1)
\]

Therefore:

\[
I = - \frac{1}{y_1 - y_0} \cos(y_1) + \frac{1}{(y_1 - y_0)^2} \left(\sin(y_1) - \sin y_0 \right)
\]

Final Expression for the Line Integral

Putting everything together, the original line integral evaluates to:

\[
\boxed{
\int_C x \sin(y) ds = L \, x_1 \times I
}
\]

where

\[
L = \sqrt{x_1

Frequently Asked Questions

What is the general approach to evaluate the line integral of C x sin(y) ds over a line segment from (0,0) to (a,b)?

To evaluate the line integral of C x sin(y) ds over the segment from (0,0) to (a,b), parameterize the curve, compute the derivatives to find ds, substitute into the integrand, and integrate over the parameter interval.

How do you parameterize the line segment from (0,0) to (a,b)?

A common parameterization is r(t) = (at, bt) for t in [0,1], where x = a t and y = b t, ensuring the curve runs from (0,0) to (a,b).

What is the expression for ds in terms of the parameter t?

Since r(t) = (a t, b t), then dr/dt = (a, b), and ds = $\sqrt{(a^2 + b^2)}$ dt.

How do you set up the integral for the line integral of C x sin(y) ds?

The integral becomes $\int_0^1 (a\,t) \sin(b\,t) \sqrt{a^2 + b^2}\, dt$, integrating with respect to t from 0 to 1.

What techniques are useful for evaluating the resulting integral?

Techniques such as substitution or integration by parts can be useful, especially if the integral involves products like $t \sin(b\,t)$.

Can the integral be simplified before integrating?

Yes, factor out constants like $\sqrt{a^2 + b^2}$ and rewrite the integral as $(a \sqrt{a^2 + b^2}) \int_0^1 t \sin(b\,t)\, dt$ to make it easier to evaluate.

What is the integral of t sin(b t) dt, and how is it computed?

Use integration by parts: let $u = t$, $dv = \sin(b\,t)\, dt$; then $du = dt$, $v = -\cos(b\,t)/b$. The integral becomes $-t \cos(b\,t)/b + \int \cos(b\,t)/b\, dt$, which simplifies further.

What is the final step after computing the integral to find the value of the line integral?

Evaluate the resulting expression at the bounds $t=1$ and $t=0$, then multiply by the constants factored out earlier to obtain the final value of the line integral.

Additional Resources

Comprehensive Evaluation of the Line Integral $\int_{C} x \sin y\, ds$ Along a Given Curve C

Introduction to Line Integrals and their Significance

Line integrals are fundamental concepts in vector calculus and are instrumental in various fields such as physics, engineering, and mathematics. They allow us to evaluate the accumulation of a scalar or vector field along a specified curve within a domain. Specifically, the line integral of a scalar field $f(x,y)$ along a curve C quantifies the total "weighted" length or quantity accumulated as we traverse the path.

In the context of the integral:

$$
\int_{C} x \sin y\, ds,
$$

we are integrating the scalar function $f(x, y) = x \sin y$ along a curve C. The variable ds denotes an infinitesimal arc length element along C.

Understanding how to evaluate this integral involves analyzing the curve's parametrization, calculating the differential arc length, and integrating the resulting expression.

Fundamental Concepts and Definitions

Line Integral of a Scalar Field

Given a scalar field $f(x, y)$ and a curve C parametrized by $\mathbf{r}(t) = (x(t), y(t))$, for t in $[a, b]$, the line integral of f along C is:

$$
\int_{C} f(x, y) ds = \int_{a}^{b} f(x(t), y(t)) \left|\mathbf{r}'(t)\right| dt,
$$

where

$$
\left|\mathbf{r}'(t)\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}
$$

is the differential arc length element expressed in terms of the parametrization derivatives.

Parameterization of the Curve C

The key to evaluating the integral lies in an appropriate parameterization of the curve C. The choice of parameterization affects the ease of calculation, especially when the curve is a line segment, a circle, or any other standard geometric shape.

Specifics of the Given Integral $\int_{C} x \sin y\, ds$

1. Understanding the Function $f(x, y) = x \sin y$

This scalar field combines both the x-coordinate and the y-coordinate through a product involving the sine function. The properties of this function include:

- Linearity: The integral of f over a curve depends on how x and y vary along C.
- Boundedness: Since $\sin y$ oscillates between -1 and 1, the magnitude of f is bounded by $|x|$.

2. Nature of the Curve C

The problem states that C is a line segment from $(0, \dots)$ to some point, possibly (a, b). For a comprehensive evaluation, several specific cases can be considered:

- Line segment from $(0, y_0)$ to (x_1, y_1): General case.
- Vertical or horizontal line segments: Simplify calculations.
- Other curves: Parabolas, circles, or arbitrary curves, requiring different parametrizations.

Given the incomplete description, we will proceed with a general approach and then explore specific scenarios.

Step-by-Step Evaluation of the Line Integral

Step 1: Determine the Parametrization of C

Suppose the curve C is a line segment from point $A = (x_0, y_0)$ to point $B = (x_1, y_1)$. A straightforward parametrization is:

$$
\mathbf{r}(t) = (x(t), y(t)), \quad t \in [0, 1],
$$
where

$$
x(t) = x_0 + t (x_1 - x_0),
$$
$$
y(t) = y_0 + t (y_1 - y_0).
$$

This linear parametrization smoothly traces the line segment from A to B. The derivatives are:

$$
\frac{dx}{dt} = x_1 - x_0, \quad \frac{dy}{dt} = y_1 - y_0,
$$

and the magnitude of the derivative vector:

$$
|\mathbf{r}'(t)| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}.
$$

Step 2: Express $f(x, y)$ in Terms of t

Substitute the parametrizations:

$$
f(x(t), y(t)) = x(t) \sin y(t) = [x_0 + t (x_1 - x_0)] \sin [y_0 + t (y_1 - y_0)].
$$

This expression simplifies the integral to:

\[

$$
\int_0^1 [x_0 + t (x_1 - x_0)] \sin [y_0 + t (y_1 - y_0)] |\mathbf{r}'(t)| dt.
$$

Step 3: Set Up the Integral

The line integral becomes:

$$
\boxed{
\int_{C} x \sin y\, ds = |\mathbf{r}'(t)| \int_0^1 [x_0 + t (x_1 - x_0)] \sin [y_0 + t (y_1 - y_0)] dt.
}
$$

Note that the constant factor $|\mathbf{r}'(t)|$ can be taken outside the integral if the magnitude is constant. For general points, it may depend on t, but in the case of a straight line, it is constant because the derivatives are fixed.

Analytical Strategies for Evaluation

1. Direct Integration

When the integral simplifies to a form involving elementary functions, direct integration is feasible. For example:

$$
\int_0^1 t \sin(\alpha + \beta t) dt,
$$

which can be evaluated using integration by parts or substitution.

2. Change of Variables

Using substitution to handle the sine term:

$$
u = y_0 + t (y_1 - y_0),
$$

then:

$$
du = (y_1 - y_0) dt,
$$

which simplifies the sine integral.

Exploring Special Cases and Simplifications

Case 1: Horizontal Line Segment (y constant)

Suppose C is a horizontal line segment from (x_0, y_0) to (x_1, y_0):

- The parametrization simplifies to:

$$
x(t) = x_0 + t (x_1 - x_0), \quad y(t) = y_0,
$$

- The derivative:

$$
|\mathbf{r}'(t)| = |x_1 - x_0|,
$$

- The integral reduces to:

$$
\int_0^1 x(t) \sin y_0 \cdot |x_1 - x_0| dt = \sin y_0 |x_1 - x_0| \int_0^1 [x_0 + t (x_1 - x_0)] dt,
$$

which simplifies further to:

$$
\sin y_0 |x_1 - x_0| \left[x_0 t + \frac{(x_1 - x_0)}{2} t^2 \right]_0^1 = \sin y_0 |x_1 - x_0| \left(x_0 + \frac{x_1 - x_0}{2} \right) = \sin y_0 |x_1 - x_0| \frac{x_0 + x_1}{2}.
$$

This case is straightforward and demonstrates how the integral depends on the endpoints' coordinates and the constant y.

Case 2: Vertical Line Segment (x constant)

Similarly, for a vertical segment from (x_0, y_0) to (x_0, y_1):

- The parametrization:

$$
x(t) = x_0, \quad y(t) = y_0 + t (y_1 - y_0),
$$

- Derivative magnitude:

$$
|\mathbf{r}'(t)| = |y_1 - y_0|,
$$

- The integral:

\[
x_0 \int_0^1 \sin [y_0 + t (y_1 - y_0)] | y_1 - y_0 | dt.
\]

Change variable:

\[
u = y_0 + t (y_1 - y_0), \quad du = (y_1 - y_0) dt,
\]

then:

\[
\Rightarrow \int_{y_0}^{y_1} \sin u\, du = -\cos u \big|_{y_0}^{y_1

Evaluate The Line Integral Where C Is The Given Curve C X Sin Y Ds C Is The Line Segment From 0

Evaluate The Line Integral Where C Is The Given Curve C X Sin Y Ds C Is The Line Segment From 0

Related Articles

- Crich Corporation Uses Direct Labor-hours In Its Predetermined Overhead Rate. At The Beginning Of The
- Determine The Ratio Of The Number Of Molecules In A Gas Having A Speed Ten Times As Great As The Root
- Choose The Word, Clause, Or Phrase Modified By The Underlined Word(s).Spilled All Over Draco Malfoy's

Back to Home