Explain Why The Function Is Differentiable At The Given Point. Then Find The Linearization Lsx, Yd Of

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Introduction

Understanding the differentiability of a function at a specific point is fundamental in calculus, as it allows us to approximate the function locally by a linear function. Once differentiability is established, we can utilize the concept of linearization to approximate the function around that point efficiently. This article delves into the reasoning behind the differentiability of a function at a particular point and guides you through the process of finding its linearization.

What Does It Mean for a Function to Be Differentiable?

Before exploring why a function is differentiable at a specific point, it's essential to clarify what differentiability entails.

Differentiability at a point means that the function has a well-defined tangent (or derivative) at that point. More formally:

- The function $f(x)$ is differentiable at $x=a$ if the limit

$$
f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
$$

exists and is finite.

- Geometrically, this implies the graph of f has a tangent line at $x=a$, and the slope of this tangent line is $f'(a)$.

Key implications of differentiability:

- The function is continuous at the point a.
- The derivative $f'(a)$ exists and is finite.
- The function can be approximated locally by a linear function near a.

Criteria for Differentiability at a Given Point

To determine whether a function is differentiable at a particular point, several factors are generally considered:

1. Continuity:
Differentiability implies continuity. If the function is not continuous at the point, it cannot be differentiable there.

2. Existence of the Derivative:
The limit defining the derivative must exist. This involves analyzing the behavior of the difference quotient from both sides.

3. Smoothness of the Function:
Functions with corners, cusps, vertical tangents, or discontinuities are not differentiable at those points.

4. Analytic Approach:
For many functions, computing the derivative directly using rules of differentiation confirms differentiability at a point.

Step-by-Step Reasoning for Establishing Differentiability

1. Check Continuity at the Point

- Verify that $\lim_{x \to a}f(x) = f(a)$.

2. Compute the Difference Quotient

$$
\frac{f(a+h) - f(a)}{h}
$$

- Analyze the limit as $h \to 0$.

3. Examine the Limit

- If the limit exists and is finite, the function is differentiable at a.
- If not, it is not differentiable.

Example: Differentiability of a Sample Function

Suppose we examine the function:

$$
f(x) = \begin{cases}
x^2, & x \neq 1 \\
2, & x=1
\end{cases}
$$

\]

At $x=1$:

- Check continuity:

$$
\lim_{x \to 1} f(x) = 1^2 = 1 \neq 2 = f(1)
$$

- Since the limit does not equal the function value, f is not continuous at $x=1$. Therefore, it cannot be differentiable there.

General Conditions Ensuring Differentiability

- Smooth functions: Polynomial, exponential, logarithmic, and trigonometric functions are differentiable everywhere in their domains.

- Piecewise functions: Differentiability depends on the matching derivatives at the boundary points.

- Composite functions: Differentiability depends on the differentiability of the inner and outer functions.

Finding the Linearization L_{x_s, y_d}

Once differentiability at a point a is established, we can find the linearization (or tangent line approximation) of the function at that point. The linearization provides the best linear approximation of $f(x)$ near $x=a$.

Definition of Linearization

The linearization of a function $f(x)$ at $x=a$ is given by:

$$
L_{a}(x) = f(a) + f'(a)(x - a)
$$

This linear function approximates $f(x)$ near $x=a$.

Step-by-Step Process to Find L_{x_s, y_d}

1. Determine the Point (x_s, y_d)

- Usually, x_s is the point where the linearization is calculated.
- $y_d = f(x_s)$, the function's value at x_s.

2. Compute the Derivative at x_s

- Find $f'(x_s)$ using differentiation rules.

3. Write the Equation of the Tangent Line

$$
L_{x_s, y_d}(x) = y_d + f'(x_s)(x - x_s)
$$

This is the linear approximation of $f(x)$ near $x=x_s$.

Example: Computing the Linearization

Suppose $f(x) = \sin x$, and we want the linearization at $x_s = \pi/4$.

Step 1: Calculate $f(\pi/4)$:

$$
f(\pi/4) = \sin(\pi/4) = \frac{\sqrt{2}}{2}
$$

Step 2: Compute $f'(\pi/4)$:

$$
f'(x) = \cos x \implies f'(\pi/4) = \cos(\pi/4) = \frac{\sqrt{2}}{2}
$$

Step 3: Write the linearization:

$$
L_{\pi/4}(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \frac{\pi}{4})
$$

This linear function approximates $\sin x$ near $x=\pi/4$.

Applications and Significance

- Approximate complex functions: Linearization allows for quick estimates of function values without complex calculations.

- Analyzing behavior near points: It helps understand local behavior, slopes, and tangents.

- Solving differential equations: Linear approximations are fundamental in numerical methods.

- Optimization problems: Derivatives and linearizations assist in finding maxima, minima, and saddle points.

Summary

- A function is differentiable at a point if its derivative exists there, which requires the limit of the difference quotient to exist and the function to be continuous at that point.
- Differentiability ensures the function can be locally approximated by a tangent line.
- To find the linearization L_{x_s, y_d}, evaluate the function and its derivative at x_s, then construct the tangent line using the formula $L_{x_s, y_d}(x) = y_d + f'(x_s)(x - x_s)$.
- Linearization is a powerful tool for approximation and analysis in calculus, providing insights into the local behavior of functions.

Final Remarks

Understanding the conditions for differentiability and how to construct linearizations is a cornerstone of calculus. It bridges the gap between the exact behavior of functions and their practical, approximate representations. Mastery of these concepts enhances problem-solving skills and provides a foundation for more advanced topics in analysis, differential equations, and applied mathematics.

Frequently Asked Questions

What does it mean for a function to be differentiable at a specific point?

A function is differentiable at a point if its derivative exists at that point, meaning the function has a well-defined tangent line and no sharp corners or cusps there.

How can I determine if a function is differentiable at a given point?

You can check if the function is continuous at that point and if the limit of the difference quotient exists and is finite. If both conditions hold, the function is differentiable there.

Why is the linearization L(x, y) useful in understanding the behavior of a function near a point?

Linearization provides the best linear approximation of the function near the point, helping to understand how the function behaves locally and simplifying calculations.

How do I compute the linearization L(x, y) of a function at a given point?

To find L(x, y), evaluate the function and its partial derivatives at the point, then construct the tangent plane equation: $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$.

What role do partial derivatives play in establishing differentiability and linearization?

Partial derivatives measure the rate of change of the function with respect to each variable. If these derivatives are continuous near the point, they ensure differentiability and help form the linear approximation.

Can a function be continuous at a point but not differentiable there? Why or why not?

Yes, a function can be continuous but not differentiable at a point if it has a sharp corner, cusp, or vertical tangent, where the derivative does not exist despite continuity.

Additional Resources

Explain Why The Function Is Differentiable At The Given Point. Then Find The Linearization L(x, y) of

Understanding the differentiability of a function at a specific point is fundamental in calculus, especially when analyzing the behavior of multivariable functions. The process involves examining the function’s limit behavior, partial derivatives, and their continuity to determine if the function can be locally approximated by a linear map. Once differentiability is established, deriving the linearization (or tangent plane) provides valuable insights into the function’s local approximation near the point of interest. This guide aims to comprehensively explain why a given function is differentiable at a particular point and then methodically compute its linearization $L(x, y)$.

The Significance of Differentiability in Multivariable Calculus

Differentiability at a point extends the concept of having a well-defined tangent line in single-variable calculus to the notion of a tangent plane in multivariable calculus. When a function $f(x, y)$ is differentiable at a point (x_0, y_0):

- The function behaves "smoothly" near that point.
- It admits a linear approximation, which is crucial for understanding local behavior.
- It guarantees the continuity of the partial derivatives in a neighborhood around (x_0, y_0).

In contrast, a function can have partial derivatives at a point but still fail to be differentiable there if those derivatives are not continuous or if the limit defining the total differential does not exist uniformly.

Step 1: Confirming Differentiability at the Given Point

1.1 Understanding the Definition

A function $f(x, y)$ is differentiable at (x_0, y_0) if there exists a linear map L such that:

$$
\lim_{(h, k) \to (0, 0)} \frac{|f(x_0 + h, y_0 + k) - f(x_0, y_0) - L(h, k)|}{\sqrt{h^2 + k^2}} = 0
$$

where $L(h, k) = A h + B k$ for some constants A, B.

1.2 Conditions for Differentiability

- Existence of Partial Derivatives: $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (x_0, y_0).
- Continuity of Partial Derivatives: The partial derivatives are continuous in a neighborhood of (x_0, y_0).
- Limit of the Difference Quotient: The difference quotient approaches a linear approximation as $(h, k) \to (0, 0)$.

1.3 Practical Approach

- Calculate the partial derivatives at (x_0, y_0).
- Check whether these derivatives are continuous around the point.
- Use the definition or known theorems (like the differentiability theorem

for functions with continuous partial derivatives) to conclude differentiability.

Step 2: Computing Partial Derivatives at the Point

Suppose the function is $f(x, y)$. To find the linearization, you first need:

- $\frac{\partial f}{\partial x}$ at (x_0, y_0)
- $\frac{\partial f}{\partial y}$ at (x_0, y_0)

Calculate these derivatives using standard differentiation rules, substituting the point's coordinates.

Step 3: Establishing the Linear Approximation $L(x, y)$

Once the partial derivatives are known, the linearization (or tangent plane) at (x_0, y_0) is given by:

$$
L(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}\bigg|_{(x_0, y_0)} (x - x_0) + \frac{\partial f}{\partial y}\bigg|_{(x_0, y_0)} (y - y_0)
$$

This linear function approximates f near (x_0, y_0).

Example: Step-by-step Application

Let’s consider a concrete example to illustrate these steps:

Suppose $f(x, y) = x^2 y + e^y$, and we want to analyze the point $(1, 0)$.

Step 1: Check Differentiability at $(1, 0)$

- The function involves polynomial and exponential parts, which are smooth everywhere.
- The partial derivatives exist and are continuous everywhere, indicating f is differentiable at $(1, 0)$.

Step 2: Find Partial Derivatives

- $\frac{\partial f}{\partial x} = 2 x y$

At $(1, 0)$: $\frac{\partial f}{\partial x} = 2 \times 1 \times 0 = 0$

- $\frac{\partial f}{\partial y} = x^2 + e^y$

At $(1, 0)$: $\frac{\partial f}{\partial y} = 1^2 + e^0 = 1 + 1 = 2$

Step 3: Compute $f(1, 0)$

$$
f(1, 0) = (1)^2 \times 0 + e^{0} = 0 + 1 = 1
$$

Step 4: Write the Linearization

$$
L(x, y) = f(1, 0) + \frac{\partial f}{\partial x}|_{(1, 0)} (x - 1) + \frac{\partial f}{\partial y}|_{(1, 0)} (y - 0)
$$

$$
L(x, y) = 1 + 0 \times (x - 1) + 2 \times y = 1 + 2 y
$$

This linear approximation indicates that near $(1, 0)$, the function behaves approximately like $1 + 2 y$.

Additional Tips for Confirming Differentiability

- Use the Chain Rule judiciously for composite functions.
- Check the limit of the difference quotient directly if uncertain.
- Leverage known theorems: If all partial derivatives are continuous at a point, the function is differentiable there.
- Beware of exceptions: Functions like $f(x, y) = \frac{x y}{x^2 + y^2}$ at $(0,0)$ may have partial derivatives but are not differentiable.

Conclusion: The Power of Linearization

The process of confirming differentiability and deriving the linearization $L(x, y)$ serves as a cornerstone in multivariable calculus. It not only provides a tangible tool for approximating complex functions locally but also deepens understanding of the function’s behavior near a point. By carefully analyzing derivatives, their continuity, and limits, you ensure rigorous conclusions about differentiability and effectively construct the tangent plane approximation. Mastery of these steps enhances your analytical skills and prepares you for advanced topics in differential calculus, optimization, and differential equations.

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