Find An Equation Of The Tangent To The Curve At The Given Point By Both Eliminating The Parameter And

Find An Equation Of The Tangent To The Curve At The Given Point By Both Eliminating The Parameter And exploring methods to determine the tangent line to a curve at a specific point is fundamental in calculus. This process involves understanding parametric equations, their derivatives, and techniques to eliminate parameters to find the Cartesian form of the tangent line. Whether you're studying for exams, tackling math assignments, or simply seeking to deepen your understanding of calculus, mastering these methods is essential for analyzing curves and their properties.

Understanding the Concept of Tangent Lines to Curves

Before diving into the methods, it's important to grasp what a tangent line represents. A tangent line to a curve at a given point is the straight line that just "touches" the curve at that point without crossing it (locally). The slope of this line is equal to the derivative of the curve at that point.

In the case of parametric equations, the curve is described using a parameter, usually denoted as t. For example:

$$
x = f(t), \quad y = g(t)
$$

The challenge is to find the equation of the tangent line at a specific point (x_0, y_0) on the curve. This is often done via two methods:

1. Eliminating the Parameter to find the Cartesian form of the curve and then differentiating.
2. Using the derivatives of the parametric functions directly to find the slope at the point, then formulating the tangent line.

Method 1: Eliminating the Parameter to Find the Cartesian Equation and Then Deriving the Tangent Line

This method involves converting the parametric equations into an explicit Cartesian form, then using calculus to find the derivative and the tangent line's equation.

Step 1: Express the curve in Cartesian form

Given parametric equations:

$$
x = f(t), \quad y = g(t)
$$

Eliminate the parameter t by solving one of the equations for t and substituting into the other. For example:

- Solve $x = f(t)$ for t:

$$
t = f^{-1}(x)
$$

- Substitute into $y = g(t)$:

$$
y = g(f^{-1}(x))
$$

This yields the Cartesian form $y = h(x)$.

Note: Not all parametric equations can be easily eliminated; in some cases, implicit differentiation might be more practical.

Step 2: Find the derivative $\frac{dy}{dx}$

Once in Cartesian form, differentiate y with respect to x:

$$
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}
$$

The derivative at the specific parameter t_0 corresponding to the point (x_0, y_0) provides the slope of the tangent line.

Step 3: Find the slope at the given point

Calculate:

$$
m = \left. \frac{dy}{dx} \right|_{t = t_0}
$$

where t_0 is the parameter value at which the curve passes through (x_0, y_0). To find t_0:

$$
x_0 = f(t_0), \quad y_0 = g(t_0)
$$

Once t_0 is identified, compute m.

Step 4: Write the equation of the tangent line

Using the point-slope form:

$$
y - y_0 = m (x - x_0)
$$

This is the equation of the tangent line to the curve at (x_0, y_0).

Method 2: Using Derivatives of the Parametric Functions Directly

This method avoids explicitly eliminating the parameter, focusing instead on derivatives with respect to t.

Step 1: Compute derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$

Given:

$$
x = f(t), \quad y = g(t)
$$

Calculate:

$$
f'(t) = \frac{dx}{dt}, \quad g'(t) = \frac{dy}{dt}
$$

Step 2: Find the slope $\frac{dy}{dx}$ at the parameter t_0

The slope of the tangent line at t_0 is:

$$
\left. \frac{dy}{dx} \right|_{t = t_0} = \frac{g'(t_0)}{f'(t_0)}
$$

Ensure $f'(t_0) \neq 0$ to avoid division by zero.

Step 3: Find the coordinates (x_0, y_0) at t_0

Calculate:

$$
x_0 = f(t_0), \quad y_0 = g(t_0)
$$

Make sure that these coordinates match the given point. If they do not, revisit the choice of t_0.

Step 4: Write the tangent line equation

Using the point-slope form:

$$
y - y_0 = m (x - x_0)
$$

where $m = \frac{g'(t_0)}{f'(t_0)}$.

Worked Example: Finding the Tangent Line to a Parametric Curve

Suppose the parametric equations are:

$$
x = t^2 + 1, \quad y = 2t + 3
$$

and the point of tangency is $(x_0, y_0) = (5, 5)$.

Using Method 1: Eliminating the Parameter

Step 1: Express t in terms of x:

$$
x = t^2 + 1 \Rightarrow t^2 = x - 1
$$

$$
t = \pm \sqrt{x - 1}
$$

Step 2: Express y in terms of x:

$$
y = 2t + 3
$$

Substitute $t = \pm \sqrt{x - 1}$:

$$
y = 2 \times \pm \sqrt{x - 1} + 3
$$

This is an implicit relation, but to find the slope at the point $(5, 5)$, find the corresponding t:

$$
x_0 = 5 = t^2 + 1 \Rightarrow t^2 = 4 \Rightarrow t = \pm 2
$$

Check which t gives $y = 5$:

$$
y = 2t + 3
$$

For $t = 2$:

$$
y = 2(2) + 3 = 4 + 3 = 7 \neq 5
$$

For $t = -2$:

$$
y = 2(-2) + 3 = -4 + 3 = -1 \neq 5
$$

Neither matches $y = 5$, which indicates that the point $(5, 5)$ is not on the curve generated by these parametric equations. Therefore, the point of tangency does not lie on the curve, or the parameters do not correspond directly. For the sake of illustration, suppose the point is $(5, 7)$, which matches $t=2$.

Using Method 2: Derivatives of the Parametric Functions

Step 1: Compute derivatives:

$$
f'(t) = 2t \\
g'(t) = 2
$$

Step 2: Find t corresponding to $x=5$:

$$
x = t^2 + 1 = 5 \Rightarrow t^2 = 4 \Rightarrow t = \pm 2

$$

Step 3: Find y at $t = 2$:

$$
y = 2(2) + 3 = 7
$$

Corresponds to the point $(5, 7)$.

Step 4: Calculate the slope:

$$
m = \frac{g'(t)}{f'(t)} = \frac{2}{2t}
$$

At $t = 2$:

$$
m = \frac{2}{2 \times 2} = \frac{2}{4} = 0.5
$$

Step 5: Write the tangent line:

$$
y - 7 = 0.5 (x - 5)
$$

or

$$
y = 0.5x + 4.5
$$

Special Cases and Important Considerations

While applying these methods, be aware of certain special cases and potential pitfalls: