Find Equations Of The Tangents To The Curve X = 6t2 6, Y = 4t3 2 That Pass Through The Point (12, 6).

Find Equations Of The Tangents To The Curve X = 6t^2 + 6, Y = 4t^3 + 2 That Pass Through The Point (12, 6)

Understanding how to find the equations of tangents to a parametric curve passing through a specific point is a fundamental concept in calculus. In this article, we will explore the step-by-step process to determine the equations of the tangents to the given parametric curve $X = 6t^2 + 6$, $Y = 4t^3 + 2$, which pass through the point $(12, 6)$. This problem combines parametric equations, derivatives, and geometric insights, making it an excellent example for learners aiming to strengthen their calculus skills.

Understanding the Problem

Before diving into calculations, it's essential to understand what the problem entails:

- The curve is given parametrically by:

$$X = 6t^2 + 6$$

$$Y = 4t^3 + 2$$

- The goal is to find the equations of all tangent lines to this curve that pass through a fixed external point $(12, 6)$.

This involves two key steps:

1. Find the points on the curve where the tangents pass through $(12, 6)$.
2. Determine the equations of those tangent lines.

Step 1: Find the Slope of the Tangent at Any Point on the Curve

Since the curve is given parametrically, the slope of the tangent line at any parameter t is given by:

$$
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}
$$

Calculating the derivatives:

$$
\frac{dx}{dt} = \frac{d}{dt}(6t^2 + 6) = 12t
$$

$$
\frac{dy}{dt} = \frac{d}{dt}(4t^3 + 2) = 12t^2
$$

Therefore,

$$
\frac{dy}{dx} = \frac{12t^2}{12t} = t
$$

This is a crucial result: the slope of the tangent at parameter t is simply t.

Step 2: Equation of the Tangent Line at a Point on the Curve

The point on the curve corresponding to parameter t is:

$$
\left(X(t), Y(t) \right) = \left(6t^2 + 6, 4t^3 + 2 \right)
$$

The tangent line at this point has slope t, so its equation in point-slope form is:

$$
Y - Y(t) = t (X - X(t))
$$

or explicitly:

$$
Y - (4t^3 + 2) = t \left(X - (6t^2 + 6) \right)
$$

Step 3: Find the Tangent Lines Passing Through the Point (12, 6)

Since we want the tangent lines passing through $(12, 6)$, the point $(12, 6)$ must satisfy the tangent line equation:

$$
6 - (4t^3 + 2) = t \left(12 - (6t^2 + 6) \right)
$$

Simplify both sides:

Left side:

$$
6 - 4t^3 - 2 = 4 - 4t^3
$$

Right side:

$$
t \left(12 - 6t^2 - 6 \right) = t (6 - 6t^2) = 6t - 6t^3
$$

Set these equal:

$$
4 - 4t^3 = 6t - 6t^3
$$

Bring all terms to one side:

$$
4 - 4t^3 - 6t + 6t^3 = 0
$$

Combine like terms:

$$
(4 - 6t) + (-4t^3 + 6t^3) = 0
$$
$$
(4 - 6t) + 2t^3 = 0
$$

Rewrite:

$$
2t^3 - 6t + 4 = 0

$$

Divide entire equation by 2 for simplicity:

$$
t^3 - 3t + 2 = 0
$$

Step 4: Solve the Cubic Equation for t

We now solve:

$$
t^3 - 3t + 2 = 0
$$

This is a standard cubic equation. Possible rational roots are factors of the constant term 2:

$$
\pm 1, \pm 2
$$

Test these:

- For $t = 1$:

$$
1 - 3 + 2 = 0
$$

Yes! So, $t = 1$ is a root.

- For $t = -1$:

$$
-1 + 3 + 2 = 4 \neq 0
$$

- For $t = 2$:

$$
8 - 6 + 2 = 4 \neq 0
$$

- For $t = -2$:

$$
-8 + 6 + 2 = 0
$$

Yes! So, $t = -2$ is also a root.

Since the degree is 3 and we have two roots, we can factor the cubic as:

$$
(t - 1)(t + 2)(\text{quadratic factor})
$$

Perform polynomial division or factorization:

Divide $t^3 - 3t + 2$ by $(t - 1)(t + 2)$:

Note that:

$$
(t - 1)(t + 2) = t^2 + 2t - t - 2 = t^2 + t - 2
$$

Now, perform polynomial division of $t^3 - 3t + 2$ by $t^2 + t - 2$:

Set up division:

- Divide t^3 by t^2: quotient t

Multiply $t(t^2 + t - 2) = t^3 + t^2 - 2t$

Subtract:

$$
(t^3 - 3t + 2) - (t^3 + t^2 - 2t) = - t^2 + (-3t + 2t) + 2 = - t^2 - t + 2
$$

Next, divide $- t^2$ by t^2: quotient -1

Multiply:

$$
-1 \times (t^2 + t - 2) = - t^2 - t + 2
$$

Subtract:

$$
(- t^2 - t + 2) - (- t^2 - t + 2) = 0
$$

No remainder, so the factorization is:

\[
t^3 - 3t + 2 = (t - 1)(t + 2)(t + 1)
\]

Note that after division, the last factor is \(t + 1 \).

Thus, roots are:

\[
t = 1, \quad t = -2, \quad t = -1
\]

Step 5: Find the Corresponding Points on the Curve

For each value of \(t \), compute the point on the curve:

At \(t = 1 \):

\[
X = 6(1)^2 + 6 = 6 + 6 = 12
\]
\[
Y = 4(1)^3 + 2 = 4 + 2 = 6
\]

Point: \((12, 6)\)

At \(t = -2 \):

\[
X = 6(-2)^2 + 6 = 6 \times 4 + 6 = 24 + 6 = 30
\]
\[
Y = 4(-2)^3 + 2 = 4(-8) + 2 = -32 + 2 = -30
\]

Point: \((30, -30)\)

At \(t = -1 \):

\[
X = 6(-1)^2 + 6 = 6 + 6 = 12
\]
\[
Y = 4(-1)^3 + 2 = -4 + 2 = -2

\]

Point: \((12, -2)\)

Step 6: Write Equations of the Tangent Lines

Recall the tangent line at parameter \(t \) passes through \((X(t), Y(t))\) with slope \(t \):

\[
Y - Y(t) = t (X - X(t))
\]

For \(t = 1 \) and point \((12, 6)\):

\[
Y - 6 = 1 (X - 12) \

Frequently Asked Questions

How do we find the equations of tangents to the curve x = 6t^2 + 6, y = 4t^3 + 2 that pass through the point (12, 6)?

First, express x and y in terms of t, find dy/dx by differentiating with respect to t, then write the tangent equation at a point t and set it to pass through (12,6). Solve for t to find the specific tangents.

What is the significance of parameter t in finding the tangent lines to the given parametric curve?

Parameter t defines points on the curve, allowing us to compute derivatives and tangent equations at specific points corresponding to t-values.

How do you derive dy/dx for the parametric equations x = 6t^2 + 6 and y = 4t^3 + 2?

Differentiate x and y with respect to t: dx/dt = 12t and dy/dt = 12t^2. Then, dy/dx = (dy/dt) / (dx/dt) = t.

What is the general form of the tangent line to the curve at a point corresponding to parameter t?

The tangent line at t is given by: y - y0 = (dy/dx) (x - x0), where (x0, y0) is the point on the curve at t.

How do we determine the specific t-values for which the tangent passes through (12, 6)?

Substitute the parametric equations into the tangent line equation and impose the condition that the line passes through (12, 6), then solve for t.

Can you outline the steps to find all tangents passing through (12, 6) for this curve?

Yes. Step 1: Express x and y in terms of t. Step 2: Find dy/dx in terms of t. Step 3: Write the tangent line at t. Step 4: Impose that the line passes through (12, 6). Step 5: Solve for t to find the specific tangents.

What equations do we solve to find the tangents passing through (12, 6)?

We solve the equation obtained by substituting the parametric expressions into the tangent line equation and setting (12, 6) on that line, leading to a polynomial in t.

Are there multiple tangents passing through the point (12, 6) for this curve?

Yes, depending on the solutions for t, there may be two or more tangent lines passing through the given point.

How do the derivatives relate to the slope of the tangent lines at different points on the curve?

The derivative dy/dx at a parameter t gives the slope of the tangent line at that point on the curve.

What is the final step after solving for t to find the equations of the tangent lines passing through (12, 6)?

Plug the t-values back into the tangent line equation y - y0 = (dy/dx)(x - x0) to get the explicit equations of the tangent lines.

Additional Resources

Find equations of the tangents to the curve $x = 6t^2 + 6$, $y = 4t^3 - 2$ that pass through the point (12, 6).

This problem involves applying concepts from calculus and parametric equations to determine the equations of tangent lines to a given curve that pass through a specific external point. It offers a comprehensive exploration of how to handle parametric curves, find slopes of tangents, and relate these to external points. Let's delve into understanding the problem thoroughly and then step-by-step derive the solutions.

Understanding the Problem

In this problem, we are given a parametric curve:

$$
x = 6t^2 + 6
$$
$$
y = 4t^3 - 2
$$

and asked to find the equations of tangent lines to this curve that pass through the point (12, 6).

Key Concepts Involved:

- Parametric equations: The curve is defined in terms of a parameter t. For each t, we get a point (x, y) on the curve.
- Tangent line to a parametric curve: The slope of the tangent at a point is given by $\frac{dy}{dx}$, which can be computed via derivatives with respect to the parameter t.
- External point condition: The tangent line must pass through the point $(12, 6)$, which is outside the curve in general.

Step 1: Derivatives and Slope of the Tangent

To find the tangent line at any point on the curve, we need the slope $\frac{dy}{dx}$.

Since x and y are given in terms of t, we compute:

$$
\frac{dx}{dt} = \frac{d}{dt}(6t^2 + 6) = 12t
$$
$$
\frac{dy}{dt} = \frac{d}{dt}(4t^3 - 2) = 12t^2
$$

The slope of the tangent line at a point corresponding to parameter t is:

$$
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12t^2}{12t} = t, \quad \text{for } t \neq 0
$$

Note: When $t = 0$, $\frac{dy}{dx}$ is indeterminate because $\frac{dx}{dt} = 0$.

Step 2: Equation of the Tangent Line at a General Point

The point on the curve corresponding to parameter t is:

$$
(x_t, y_t) = (6t^2 + 6, 4t^3 - 2)
$$

The slope at this point is $m = t$. Therefore, the equation of the tangent line at this point is:

$$
y - y_t = m (x - x_t)
$$
$$
y - (4t^3 - 2) = t \left(x - (6t^2 + 6) \right)
$$

Step 3: Condition for the Line Passing Through (12, 6)

We want the tangent line to pass through the point $(12, 6)$. Substituting $(x, y) = (12, 6)$ into the tangent line equation:

$$
6 - (4t^3 - 2) = t \left(12 - (6t^2 + 6) \right)
$$

Simplify both sides:

Left side:
$$
6 - 4t^3 + 2 = 8 - 4t^3
$$

Right side:
$$
t (12 - 6t^2 - 6) = t (6 - 6t^2) = 6t - 6t^3
$$

Set the two sides equal:

$$
8 - 4t^3 = 6t - 6t^3
$$

Bring all to one side:

$$
8 - 4t^3 - 6t + 6t^3 = 0
$$

Combine like terms:

$$
(8 - 6t) + (-4t^3 + 6t^3) = 0
$$
$$
(8 - 6t) + 2t^3 = 0
$$

Rewrite:

$$
2t^3 - 6t + 8 = 0
$$

Divide through by 2 for simplicity:

$$
t^3 - 3t + 4 = 0
$$

This is a cubic equation in t:

$$
t^3 - 3t + 4 = 0
$$

Step 4: Solving the Cubic Equation

Let's solve:

$$
t^3 - 3t + 4 = 0
$$

Rational Root Test:

Possible rational roots are factors of 4 over factors of 1: $\pm1, \pm2, \pm4$.

Test $t = 1$:

$$
1 - 3 + 4 = 2 \neq 0
$$

Test $t = -1$:

$$
-1 + 3 + 4 = 6 \neq 0
$$

Test $t= 2$:

$$
8 - 6 + 4 = 6 \neq 0
$$

Test $t= -2$:

$$
-8 + 6 + 4 = 2 \neq 0
$$

Test $t= 4$:

$$
64 - 12 + 4 = 56 \neq 0
$$

Test $t= -4$:

$$
-64 + 12 + 4 = -48 \neq 0
$$

No rational roots are found among these candidates. Therefore, roots are irrational or complex. We proceed with the cubic formula or numerical methods.

Step 5: Finding Roots Using Cardano's Method

Given the cubic:

$$
t^3 - 3t + 4 = 0
$$

Express in depressed cubic form:

$$
t^3 + pt + q = 0
$$
where $p = -3$, $q= 4$.

The discriminant:

$$
\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = \left(2\right)^2 + \left(-1\right)^3 = 4 + (-1) = 3 > 0
$$

Since $\Delta > 0$, there is one real root and two complex conjugate roots.

The real root:

$$
t = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}}
$$

Compute:

$$
-\frac{q}{2} = -2
$$
$$
\sqrt{\Delta} = \sqrt{3} \approx 1.732
$$

Then:

$$
t = \sqrt[3]{-2 + 1.732} + \sqrt[3]{-2 - 1.732}
$$
$$
t = \sqrt[3]{-0.268} + \sqrt[3]{-3.732}
$$

Calculate cube roots:

$$
\sqrt[3]{-0.268} \approx -0.65
$$
$$
\sqrt[3]{-3.732} \approx -1.56
$$

Sum:

$$
t \approx -0.65 - 1.56 = -2.21
$$

So, the real root approximately:

$$
t \approx -2.21
$$

And the complex roots can be found using the quadratic formula for the remaining roots, but for the purpose of the tangent lines, the real roots are most relevant.

Step 6: Find the Corresponding Points and Equations

For $t \approx -2.21$:

Calculate the point:

$$
x = 6t^2 + 6
$$
$$
x \approx 6 \times (-2.21)^2 + 6 = 6 \times 4.88 + 6 \approx 29.28 + 6 = 35.28
$$

$$
y = 4t^3 - 2
$$
$$
t^3 \approx (-2.21)^3 \approx -10.8
$$
$$
y \approx 4 \times (-10.8) - 2 = -43.2 - 2 = -45.2
$$

The slope at this point:

$$
m = t \approx -2.21
$$

Equation of the tangent line:

$$
y - y_t = m (x - x_t)
$$
$$
y + 45.2 = -2.21 (x - 35.28)
$$

\]

Or in slope-intercept form:

\[
y = -2.21

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