Find Power Series Solution For The ODE About X = 0 In The Form Of Y= [infinity] N=0 CnX^ny - (x+1)y -

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Understanding how to find power series solutions for differential equations (ODEs) is a fundamental aspect of mathematical analysis, especially in fields like physics, engineering, and applied mathematics. When the differential equation involves a variable coefficient or is difficult to solve using elementary functions, the power series method provides a systematic approach for finding solutions around a regular point, such as x = 0. In this article, we will explore the process of deriving the power series solution for a specific ordinary differential equation about x = 0, expressed in the form:

$$ Y = \sum_{n=0}^{\infty} C_n x^n $$

and analyze how to handle the terms involved in the equation, including the term $(x+1)y$.

Understanding the Differential Equation

Before diving into the solution process, it is essential to understand the structure of the differential equation under consideration. The general form that we will work with resembles:

$$ y'' + p(x) y' + q(x) y = 0 $$

or a similar form involving the function y, its derivatives, and the variable x. The given expression appears to involve the term $(x+1) y$, indicating the differential equation might be of the form:

$$ y'' - (x+1) y = 0 $$

or a related form.

Key features:

- The point $x = 0$ is a regular point or a point where the series expansion can be centered.

- The solution y is expressed as a power series expansion around $x = 0$.

Formulating the Power Series Solution

The core idea of the power series method is to assume that the solution y can be written as:

$$y(x) = \sum_{n=0}^{\infty} C_n x^n$$

where C_n are constants to be determined, and the series converges in some neighborhood around $x = 0$.

Step 1: Deriving y' and y''

Calculating derivatives term-by-term:

$$y'(x) = \sum_{n=1}^{\infty} n C_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

Step 2: Substituting into the differential equation

Suppose the differential equation is:

$$y'' - (x+1) y = 0$$

Substitute the series expressions:

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - (x+1) \sum_{n=0}^{\infty} C_n x^n = 0$$

Step 3: Simplify each term

- For the first sum:

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

- For the second term:

$$(x+1) y = x y + y$$

which expands to:

$$ x y = x \sum_{n=0}^\infty C_n x^n = \sum_{n=0}^\infty C_n x^{n+1} $$

and

$$ y = \sum_{n=0}^\infty C_n x^n $$

Therefore,

$$ (x+1) y = \sum_{n=0}^\infty C_n x^{n+1} + \sum_{n=0}^\infty C_n x^n $$

Gathering Like Terms and Forming the Recurrence Relation

To equate the series, align the powers of x in all sums.

Step 1: Reindex the sums to match powers

- For the first sum involving y'':

Let $k = n - 2 \Rightarrow n = k + 2$

$$ y'' = \sum_{k=0}^\infty (k+2)(k+1) C_{k+2} x^{k} $$

- For the term $\sum_{n=0}^\infty C_n x^{n+1}$:

Let $m = n + 1 \Rightarrow n = m -1$

$$ \sum_{m=1}^\infty C_{m-1} x^{m} $$

- For the term $\sum_{n=0}^\infty C_n x^{n}$, it stays as is.

Step 2: Write the entire sum with aligned powers

The differential equation becomes:

$$
\sum_{k=0}^\infty (k+2)(k+1) C_{k+2} x^{k} - \left(\sum_{m=1}^\infty C_{m-1} x^{m} + \sum_{n=0}^\infty C_n x^{n} \right) = 0
$$

\]

Express all sums in terms of a common index:

- For the second sum:

$$
\sum_{m=1}^\infty C_{m-1} x^{m}
$$

- For the third sum:

$$
\sum_{n=0}^\infty C_n x^{n}
$$

Let’s write all sums in terms of n:

$$
\sum_{n=0}^\infty (n+2)(n+1) C_{n+2} x^{n} - \sum_{n=1}^\infty C_{n-1} x^{n} - \sum_{n=0}^\infty C_n x^{n} = 0
$$

Note that the sums over n are valid for all $n \geq 0$, with the exception of the second sum starting at $n=1$. To combine all sums into a single sum from $n=0$, we adjust the sums accordingly.

Deriving the Recurrence Relation

Now, write the entire expression as a single sum:

$$
\sum_{n=0}^\infty \left[(n+2)(n+1) C_{n+2} - C_{n-1} - C_n \right] x^n = 0
$$

where, for $n=0$:

- The term involving C_{-1} does not exist, so its coefficient is zero.

Thus, the recurrence relation is:

$$
(n+2)(n+1) C_{n+2} = C_{n-1} + C_n
$$

with the understanding that for $n=0$, $C_{-1} = 0$.

Step 1: Initial conditions

To fully determine the series, initial conditions such as C_0 and C_1 are needed. These are usually given or chosen based on boundary conditions.

Step 2: Recursion for coefficients

The recurrence relation allows calculating higher coefficients from known initial coefficients:

$$
C_{n+2} = \frac{C_{n-1} + C_n}{(n+2)(n+1)}
$$

for $n \geq 0$.

Solving for the Coefficients

Using the recurrence relation, the coefficients C_n can be computed step-by-step:

1. Choose initial values C_0 and C_1.
2. Compute C_2 using $n=0$:

$$
C_2 = \frac{C_{-1} + C_0}{2 \times 1} = \frac{0 + C_0}{2}
$$

3. Compute C_3 using $n=1$:

$$
C_3 = \frac{C_0 + C_1}{3 \times 2}
$$

4. Continue iteratively for higher n.

The general pattern indicates that the solution involves two arbitrary constants C_0 and C_1, consistent with the second-order nature of the differential equation.

Constructing the Power Series Solution

Having obtained the coefficients, the general solution around $x=0$ is:

$$
y(x) = C_0 \left(1 + \frac{x^2}{2!} + \cdots \right) + C_1 \left(x + \frac{C_0 + C_1}{3 \times 2} x^3 + \cdots \right)
$$

which can be expressed as:

$$
y(x) = \sum_{n=0}^\infty C_{2n} x^{2n} + \sum_{n=0}^\infty C_{2n+1} x^{2n+1}
$$

where the coefficients are determined via the recurrence relation.

Interpreting the Solution and Its Convergence

The power series solution provides an explicit

Frequently Asked Questions

What is the general approach to finding a power series solution for the differential equation (x+1)y" - (x+1)y = 0 about x = 0?

The typical method involves assuming a power series solution $y = \Sigma Cn x^n$, substituting into the differential equation, and then equating coefficients of like powers to find recurrence relations for Cn.

How do you set up the power series solution $y = \Sigma C_n x^n$ for the given ODE?

Replace y, y', and y'' with their power series representations, differentiate term-by-term, and substitute back into the original differential equation to derive equations for the coefficients C_n.

What is the significance of the point $x = 0$ in finding the power series solution?

Since the series is expanded about $x = 0$, the point is a regular point or a point of expansion, allowing a straightforward power series approach without concerns of convergence issues at this point.

How do the coefficients Cn relate to the recurrence relation derived from the differential equation?

The substitution leads to a recurrence relation linking C_n to previous coefficients, enabling recursive calculation of all coefficients once initial conditions are specified.

What is the form of the solution once the power series coefficients are determined?

The solution is expressed as $y = \Sigma C_n x^n$, where the coefficients C_n are obtained from the recurrence relation, providing a local power series solution around $x = 0$.

Can the power series solution be used to analyze the behavior of solutions near singular points?

Yes, but special care is needed if $x = 0$ is a singular point; in such cases, Frobenius' method may be required instead of a simple power series expansion.

How does the term $(x+1)y$ in the differential equation influence the structure of the power series solution?

The term introduces additional polynomial factors that affect the coefficients' recurrence relations, influencing the convergence and form of the power series solution.

Additional Resources

Power Series Solution for the Differential Equation About $x = 0$: An In-Depth Analysis

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Introduction

Differential equations are fundamental to understanding a wide array of phenomena in physics, engineering, biology, and many other scientific domains. Among the various techniques for solving differential equations, the power series method stands out as a powerful tool, especially for equations that are difficult or impossible to solve using elementary functions. This method involves expressing the solution as an infinite series, typically centered around a point of interest—in this case, $x=0$.

This review delves into the process of finding a power series solution for a specific ordinary differential equation (ODE):

$$
\boxed{
\text{Find a power series solution for the ODE about } x=0 \text{ in the form } y = \sum_{n=0}^{\infty} C_n x^n \text{, with the differential equation:}
}
$$

$$
\quad (x+1) y'' + (x+1) y' - y = 0
$$

(Note: The original prompt appears to have a fragmentary equation; for clarity, we interpret the ODE as a common form suitable for power series solutions, which resembles $(x+1) y'' + (x+1) y' - y = 0$. If the original was different, please specify. Here, we proceed with this form for demonstration.)

Understanding the Differential Equation

The Structure of the Equation

The given differential equation:

$$
(x+1) y'' + (x+1) y' - y = 0
$$

has notable features:

- Variable coefficients: The coefficients depend on x, making the solution less straightforward than

constant coefficient equations.
- Singularity at $x = -1$: Since the coefficients involve $x+1$, the point $x = -1$ is a potential singular point. Our primary interest is around $x=0$, which is a regular point for the equation.

Significance of the Point $x=0$

- The point $x=0$ is a regular point because the coefficient functions are analytic (polynomial) at $x=0$.
- This makes the power series method applicable: we can assume a solution in the form of a power series centered at $x=0$.

The Power Series Method: Conceptual Foundations

Fundamental Idea

Suppose the solution $y(x)$ can be expressed as an infinite power series:

$$
y(x) = \sum_{n=0}^\infty C_n x^n
$$

where:

- C_n are constants to be determined.
- The series converges within some radius of convergence, dependent on the nature of the coefficients and singularities.

Derivatives of the Power Series

To substitute into the differential equation, we need the derivatives:

$$
y'(x) = \sum_{n=1}^\infty n C_n x^{n-1}
$$

$$
y''(x) = \sum_{n=2}^\infty n(n-1) C_n x^{n-2}
$$

It's often convenient to reindex these sums to write all series in terms of x^n.

Step-by-Step Solution Approach

1. Rewrite the derivatives with aligned powers of x

- For $y''(x)$:

$$
y''(x) = \sum_{n=0}^\infty (n+2)(n+1) C_{n+2} x^{n}
$$

by setting $n \to n+2$.

- For $y'(x)$:

$$
y'(x) = \sum_{n=0}^\infty (n+1) C_{n+1} x^{n}
$$

by setting $n \to n+1$.

2. Express the entire ODE in series form

Substituting these into the original equation:

$$
(x+1) y'' + (x+1) y' - y = 0
$$

becomes:

$$
(x+1) \sum_{n=0}^\infty (n+2)(n+1) C_{n+2} x^{n} + (x+1) \sum_{n=0}^\infty (n+1) C_{n+1} x^{n} - \sum_{n=0}^\infty C_n x^{n} = 0
$$

3. Expand $(x+1)$ terms

Multiplying each series by $(x+1)$:

\[
\begin{aligned}
&(x+1) \sum_{n=0}^\infty (n+2)(n+1) C_{n+2} x^{n} \\
&= \sum_{n=0}^\infty (n+2)(n+1) C_{n+2} x^{n+1} + \sum_{n=0}^\infty (n+2)(n+1) C_{n+2} x^{n}

\end{aligned}
\]

Similarly for the \(y'\) term:

\[
(x+1) \sum_{n=0}^\infty (n+1) C_{n+1} x^{n} = \sum_{n=0}^\infty (n+1) C_{n+1} x^{n+1} + \sum_{n=0}^\infty (n+1) C_{n+1} x^{n}
\]

Now, the entire equation becomes:

\[
\left[\sum_{n=0}^\infty (n+2)(n+1) C_{n+2} x^{n} + \sum_{n=0}^\infty (n+2)(n+1) C_{n+2} x^{n+1} \right] + \left[\sum_{n=0}^\infty (n+1) C_{n+1} x^{n} + \sum_{n=0}^\infty (n+1) C_{n+1} x^{n+1} \right] - \sum_{n=0}^\infty C_n x^{n} = 0
\]

Collecting Like Terms

To derive a recurrence relation for the coefficients \(C_n\), it is essential to express all sums with the same power of \(x^n\). Let's shift indices where necessary:

- For the sums involving \(x^{n+1}\), replace \(n \to n-1\):

\[
\sum_{n=1}^\infty (n+1) C_{n+1} x^{n}
\]

- For the sums involving \(x^{n}\), they stay as-is.

Now, the entire expression becomes:

\[
\sum_{n=0}^\infty \left[(n+2)(n+1) C_{n+2} + (n+1) C_{n+1} - C_n \right] x^{n} + \sum_{n=1}^\infty \left[(n+1)(n) C_{n} + (n) C_{n} \right] x^{n}
\]

But notice that the second sum involves the same powers as the first sum. Therefore, the coefficient of \(x^n\) (for \(n \geq 1\)) becomes:

$$
(n+2)(n+1) C_{n+2} + (n+1) C_{n+1} + (n+1) C_{n+1} - C_n
$$

which simplifies to:

$$
(n+2)(n+1) C_{n+2} + 2(n+1) C_{n+1} - C_n
$$

For $n=0$, the sum reduces to:

$$
(2)(1) C_2 + (1) C_1 - C_0
$$

Recurrence Relation for Coefficients

The key step is setting the entire series sum to zero:

$$
\sum_{n=0}^\infty \left[(n+2)(n+1) C_{n+2} + 2(n+1) C_{n+1} - C_n \right] x^{n} = 0
$$

Since the series equals zero for all x, each coefficient must be zero:

$$
(n+2)(n+1) C_{n+2} + 2(n+1) C_{n+1} - C_n = 0, \quad \text{for } n=0,1,2,\dots
$$

This gives a recurrence relation:

$$
C_{n+2} = \frac{C_n - 2(n+1) C_{n+1}}{(n+2)(n+1)}
$$

Initial Conditions and Series Solution

To fully determine the solution, initial conditions C_0 and C_1 are needed, which depend on the specific problem's initial or boundary conditions.

- Given initial conditions: \(y(0) = y_0\), \(y'(

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