Find The General Solution Of The Differential Equation (x^2 + 1)tan Y Dy/dx = X. (a) Y = C/squareroot

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Understanding and solving differential equations is a fundamental aspect of advanced mathematics, with applications spanning physics, engineering, and other scientific disciplines. In this article, we focus on a specific type of differential equation:

$$
(x^2 + 1) \tan Y \frac{dy}{dx} = x
$$

Our goal is to find its general solution, especially under the condition $Y = \frac{C}{\sqrt{x}}$. Through systematic steps, substitution methods, and integration techniques, we will explore how to approach and solve this equation comprehensively.

Introduction to the Differential Equation

The differential equation in question is:

$$
(x^2 + 1) \tan Y \frac{dy}{dx} = x
$$

This is a first-order differential equation involving a nonlinear term $\tan Y$. The equation can be viewed as a relationship between the dependent variable Y and the independent variable x, where the derivative dy/dx is multiplied by a function involving x and Y.

Key features of this differential equation include:

- Nonlinear dependence on Y via $\tan Y$.
- The coefficient $(x^2 + 1)$, which is always positive.
- The right-hand side is a simple function x.

Understanding the Structure of the Equation

Before attempting to solve, it's essential to analyze the structure:

- The equation resembles a separable form if we can express it in terms of Y and x separately.
- The presence of $\tan Y$ suggests that substitution involving the tangent function might simplify the problem.
- The form hints at a substitution $Z = \tan Y$, which could linearize or simplify the equation.

Methodology for Solving the Differential Equation

The solution approach involves several key steps:

1. Substitution to Simplify the Equation

2. Rearrangement to a Separable Form

3. Integration of Both Sides

4. Applying Initial Conditions or Given Conditions

5. Expressing the General Solution

Let's explore these steps in detail.

Step 1: Substitution to Simplify $\tan Y$

Since $\tan Y$ appears prominently, define a new variable:

$$
Z = \tan Y
$$

Then, using the derivative chain rule:

$$
\frac{dy}{dx} = \frac{1}{\sec^2 Y} \frac{dZ}{dx}
$$

But this adds complexity, so a more straightforward approach is to directly rewrite the original differential equation in terms of Z:

$$
(x^2 + 1) Z \frac{dy}{dx} = x
$$

Recall that:

$$
Y = \arctan Z
$$

Differentiating both sides with respect to x:

$$
\frac{dY}{dx} = \frac{1}{1 + Z^2} \frac{dZ}{dx}
$$

Express dy/dx in terms of Z:

$$
\frac{dy}{dx} = \frac{1}{1 + Z^2} \frac{dZ}{dx}
$$

Substitute into the original equation:

$$
(x^2 + 1) Z \cdot \frac{1}{1 + Z^2} \frac{dZ}{dx} = x
$$

Step 2: Formulating the Equation in Terms of Z

Rearranging:

$$
\frac{(x^2 + 1) Z}{1 + Z^2} \frac{dZ}{dx} = x
$$

Expressed as:

$$
\frac{dZ}{dx} = \frac{x (1 + Z^2)}{(x^2 + 1) Z}
$$

This form is more manageable because it separates variables Z and x:

$$
\frac{dZ}{(1 + Z^2)/Z} = \frac{x}{x^2 + 1} dx
$$

Note that:

$$
\frac{1 + Z^2}{Z} = Z + \frac{1}{Z}
$$

Therefore, the differential equation becomes:

$$
\frac{dZ}{Z + \frac{1}{Z}} = \frac{x}{x^2 + 1} dx
$$

Simplify the left side:

\[

$$
\frac{dZ}{Z + \frac{1}{Z}} = \frac{dZ}{\frac{Z^2 + 1}{Z}} = \frac{Z}{Z^2 + 1} dZ
$$

Thus, the equation reduces to:

$$
\frac{Z}{Z^2 + 1} dZ = \frac{x}{x^2 + 1} dx
$$

Step 3: Integration of Both Sides

Now, integrate both sides:

$$
\int \frac{Z}{Z^2 + 1} dZ = \int \frac{x}{x^2 + 1} dx
$$

Left Integral:

Let’s evaluate:

$$
I_1 = \int \frac{Z}{Z^2 + 1} dZ
$$

Use substitution:

- Set $u = Z^2 + 1$,

- Then $du = 2Z dZ$,
- So, $Z dZ = \frac{1}{2} du$.

Rewriting:

$$
I_1 = \int \frac{Z}{Z^2 + 1} dZ = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln |u| + C_1
$$

Back to Z:

$$
I_1 = \frac{1}{2} \ln |Z^2 + 1| + C_1
$$

Right Integral:

Similarly,

$$
I_2 = \int \frac{x}{x^2 + 1} dx
$$

Let $v = x^2 + 1$, then $dv = 2x dx$, so:

$$
x dx = \frac{1}{2} dv
$$

Thus,

$$
I_2 = \frac{1}{2} \int \frac{1}{v} dv = \frac{1}{2} \ln |v| + C_2 = \frac{1}{2} \ln |x^2 + 1| + C_2
$$

Step 4: Combining the Results

The integrated form is:

$$
\frac{1}{2} \ln |Z^2 + 1| = \frac{1}{2} \ln |x^2 + 1| + C
$$

or

$$
\ln |Z^2 + 1| = \ln |x^2 + 1| + 2C
$$

Let $K = e^{2C}$, a constant, then:

$$
|Z^2 + 1| = K |x^2 + 1|
$$

Since $Z = \tan Y$, substitute back:

$$
(\tan Y)^2 + 1 = K (x^2 + 1)

$$
(\tan Y)^2 + 1 = \sec^2 Y
$$

Therefore:

$$
\sec^2 Y = K (x^2 + 1)
$$

Step 5: Expressing the General Solution

From the above, we have:

$$
\sec^2 Y = K (x^2 + 1)
$$

Taking square roots:

$$
\sec Y = \pm \sqrt{K} \sqrt{x^2 + 1}
$$

Expressing Y:

$$
Y = \pm \arccos \left(\frac{1}{\sqrt{K} \sqrt{x^2 + 1}} \right) + n\pi
$$

where n is an integer, accounting for the periodicity of $\arccos$.

Alternatively, the solution can be written in terms of tangent:

$$
\tan Y = \pm \sqrt{K (x^2 + 1) - 1}
$$

Since the problem specifies a particular solution form:

Given Condition:

$$
Y = \frac{C}{\sqrt{x}}
$$

we can relate the constant K to C to satisfy this initial condition or particular solution form.

Special Case: Y = C / $\sqrt{x}$

Suppose $Y = \frac{C}{\sqrt{x}}$:

- The general solution involves $\sec^2 Y$, which becomes:

$$
\sec^2 \left(\frac{C}{\sqrt{x}} \right) = K (x^2 + 1)
$$

-

Frequently Asked Questions

What is the general solution of the differential equation (x^2 + 1) tan Y dy/dx = x?

The general solution is Y = arcsin(C / √(x^2 + 1)), where C is an arbitrary constant.

How do you solve the differential equation (x^2 + 1) tan Y dy/dx = x?

By rewriting as dy/dx = x / [(x^2 + 1) tan Y], then separating variables and integrating both sides to find Y in terms of x.

What substitution can be used to solve the differential equation involving tan Y?

Using u = sin Y simplifies the equation because tan Y = sin Y / cos Y, allowing for substitution and easier integration.

What is the particular solution when the constant C = 0 in the general solution?

When C = 0, the solution reduces to Y = 0 or Y = nπ, depending on the context, representing constant solutions.

Is the differential equation separable, and how does that help in solving it?

Yes, it is separable after substitution, allowing the variables to be separated and integrated independently.

How does the initial condition Y = C / □(x^2 + 1) relate to the differential equation?

It represents the general solution form, where C is determined by initial or boundary conditions.

Can the given differential equation be solved directly without substitution?

No, substitution (such as u = sin Y) simplifies the equation, making it solvable via standard integration methods.

What is the significance of the constant C in the solution?

C represents an arbitrary constant arising from integration, capturing the family of solutions to the differential equation.

How does the form Y = C / □(x^2 + 1) relate to inverse trigonometric functions?

It indicates that the solution involves inverse sine functions, since Y = arcsin(C / □(x^2 + 1)).

What are the key steps to verify the solution Y = C / □(x^2 + 1)?

Differentiate Y with respect to x, substitute into the original differential equation, and check if both sides are equal.

Additional Resources

Comprehensive Analysis and Solution of the Differential Equation $(x^2 + 1) \tan Y \frac{dy}{dx} = x$

Introduction

Differential equations play a critical role in modeling real-world phenomena across physics, engineering, biology, and many other scientific disciplines. They describe the relationship between functions and their derivatives, providing insights into how systems evolve over time or space. Among these, first-order differential equations are the most common and foundational.

In this discussion, we focus on a specific first-order differential equation:

$$
(x^2 + 1) \tan Y \frac{dy}{dx} = x
$$

with an interest in finding its general solution. The problem also suggests a particular form of the solution, namely:

$$
Y = \frac{C}{\sqrt{\dots}}
$$

which hints at a solution involving square roots or similar expressions.

Understanding the Differential Equation

Expression Breakdown

The given differential equation is:

$$
(x^2 + 1) \tan Y \frac{dy}{dx} = x
$$

Let's analyze the components:

- $x^2 + 1$: A quadratic in x, always positive, which acts as a coefficient.
- $\tan Y$: A nonlinear function of Y.
- $\frac{dy}{dx}$: The derivative of Y with respect to x.

The presence of $\tan Y$ indicates nonlinearity, and the structure suggests that a substitution involving $\tan Y$ could be beneficial.

Goal

Our primary objective is to find the general solution $Y(x)$ that satisfies the differential equation. This involves:

- Recognizing the type of differential equation.
- Applying suitable substitutions to reduce it to a more solvable form.
- Integrating to obtain the explicit solution.

Classifying the Differential Equation

Is it Separable?

To determine whether the equation is separable, rewrite it:

$$
\frac{dy}{dx} = \frac{x}{(x^2 + 1) \tan Y}
$$

If we can express the right side as a product of a function of x and a function of Y, then the equation is separable.

Indeed,

$$
\frac{dy}{dx} = \frac{x}{(x^2 + 1)} \cdot \frac{1}{\tan Y}
$$

which can be written as:

$$
\frac{dy}{dx} = \left(\frac{x}{x^2 + 1} \right) \cot Y
$$

Now, observe that:

$$
\frac{dy}{dx} = \left(\frac{x}{x^2 + 1} \right) \cot Y
$$

This form suggests potential for separation of variables if we consider $\cot Y$ as a function of Y, and the other as a function of x.

Transforming the Differential Equation

Substitution Strategy

Since $\cot Y = \frac{1}{\tan Y}$, and the original differential equation involves $\tan Y$, a natural substitution is:

$$
Z = \tan Y
$$

which implies:

$$
Y = \arctan Z
$$

and:

$$
\frac{dy}{dx} = \frac{d}{dx} (\arctan Z) = \frac{1}{1 + Z^2} \frac{dZ}{dx}
$$

Rewriting the Differential Equation

Substitute into the original equation:

$$
(x^2 + 1) Z \cdot \frac{1}{1 + Z^2} \frac{dZ}{dx} = x
$$

\]

which simplifies to:

\[
\frac{(x^2 + 1) Z}{1 + Z^2} \frac{dZ}{dx} = x
\]

Rearranged as:

\[
\frac{dZ}{dx} = \frac{x (1 + Z^2)}{(x^2 + 1) Z}
\]

This is a crucial step because now the differential equation is expressed entirely in terms of \(Z\) and \(x\).

Recognizing the Type of Equation

Rational Form

The equation:

\[
\frac{dZ}{dx} = \frac{x (1 + Z^2)}{(x^2 + 1) Z}
\]

is a rational differential equation.

Potential for Variable Substitution

The structure suggests that separating variables might be feasible, or perhaps transforming it into an exact differential or a Bernoulli-type equation.

Separating Variables

Express as:

$$
\frac{dZ}{(1 + Z^2)} \cdot Z = \frac{x}{x^2 + 1}
$$

which leads to:

$$
\frac{Z}{1 + Z^2} dZ = \frac{x}{x^2 + 1} dx
$$

This is promising because both sides are functions of only one variable, allowing for straightforward integration.

Integrating Both Sides

Left Side: $\int \frac{Z}{1 + Z^2} dZ$

- Recognize that:

$$
\int \frac{Z}{1 + Z^2} dZ
$$

can be solved via substitution:

Let:

$$
u = 1 + Z^2 \Rightarrow du = 2Z dZ
$$

then:

$$
Z dZ = \frac{1}{2} du
$$

So,

$$
\int \frac{Z}{1 + Z^2} dZ = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |1 + Z^2| + C
$$

Right Side: $\int \frac{x}{x^2 + 1} dx$

- Recognize that:

$$
\int \frac{x}{x^2 + 1} dx
$$

\]

is a standard integral:

$$
\frac{1}{2} \ln |x^2 + 1| + C
$$

Combining the Results

The integrated form becomes:

$$
\frac{1}{2} \ln |1 + Z^2| = \frac{1}{2} \ln |x^2 + 1| + C
$$

Multiply through by 2:

$$
\ln |1 + Z^2| = \ln |x^2 + 1| + 2C
$$

Let $K = e^{2C}$, a positive constant, then:

$$
|1 + Z^2| = K |x^2 + 1|
$$

or equivalently,

$$
1 + Z^2 = A (x^2 + 1)
$$

where $A = \pm K$ is an arbitrary positive or negative constant.

Reverting to Y

Recall:

$$
Z = \tan Y
$$

and:

$$
1 + Z^2 = \sec^2 Y
$$

Thus:

$$
\sec^2 Y = A (x^2 + 1)
$$

or:

\[

$$
\boxed{\sec Y = \pm \sqrt{A (x^2 + 1)}}
$$

Taking reciprocal, we get:

$$
\cos Y = \pm \frac{1}{\sqrt{A (x^2 + 1)}}
$$

The general solution for Y is then:

$$
Y = \arccos \left(\pm \frac{1}{\sqrt{A (x^2 + 1)}} \right)
$$

where A is an arbitrary constant determined by initial conditions.

Expressing the Solution in the Form $Y = \frac{C}{\sqrt{\dots}}$

Given the form provided:

$$
Y = \frac{C}{\sqrt{\dots}}
$$

and noting that the solution involves inverse cosine functions, it suggests that the constant C could relate to $\arccos$ or $\arctan$ functions.

Alternatively, expressing Y explicitly as:

$$
Y = \arccos \left(\frac{1}{\sqrt{A (x^2 + 1)}} \right)
$$

or

$$
Y = \arccos \left(\frac{1}{\sqrt{\frac{1}{A} (x^2 + 1)}} \right)
$$

depending on the constant's manipulation.

Final Form of the General Solution

Putting it all together, the general solution is:

$$
\boxed{
Y(x) = \arccos \left(\frac{1}{\sqrt{A (x^2 + 1)}} \right)
}
$$

where A is an arbitrary constant that can be positive or negative, encapsulating initial conditions.

Alternatively, if desired, the solution can be written as:

$$
Y(x) = \arccos \left(\frac{1}{\sqrt{\frac{1}{A} (x^2 + 1)}} \right)
$$

which reflects the same relationship with

Find The General Solution Of The Differential Equation X 2 1 Tan Y Dy Dx X A Y C Squareroot

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