Find The General Solution Of The Following Equation. Express The Solution Explicitly As A Function Of

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When confronted with differential equations, one of the primary objectives is to find the general solution, which encompasses all possible solutions to the equation. The phrase "Express The Solution Explicitly As A Function Of" indicates that our goal is to derive a formula for the dependent variable explicitly in terms of the independent variable and arbitrary constants. This process often involves identifying the type of differential equation (ordinary or partial), classifying it (linear, nonlinear, separable, exact, etc.), and then applying the appropriate solving techniques.

In this comprehensive guide, we will explore various common types of differential equations, the methods used to solve them, and how to express their solutions explicitly as functions of the independent variable. Whether you are dealing with first-order ordinary differential equations (ODEs) or higher-order equations, understanding the systematic approach will help you find the general solutions effectively.

Understanding the Types of Differential Equations

Before diving into solving methods, it is essential to classify the differential equations based on their structure and order.

First-Order Differential Equations

These are equations involving derivatives of the first order only:
- Separable Equations: Can be written as $\frac{dy}{dx} = f(x)g(y)$.
- Linear Equations: Have the form $\frac{dy}{dx} + P(x)y = Q(x)$.
- Exact Equations: Satisfy the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for equations of the form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$.
- Homogeneous Equations: Can be transformed into separable equations via substitution.

Higher-Order Differential Equations

Involving derivatives of order two or higher, such as:
- Second-order linear equations like $y'' + p(x)y' + q(x)y = r(x)$.
- Homogeneous or nonhomogeneous equations.

Methods for Finding the General Solution

The solution method depends heavily on the type of differential equation. Here, we break down some common techniques.

Solving Separable Differential Equations

Separable equations can be written as:
$$ \frac{dy}{dx} = f(x)g(y) $$

Steps:
1. Rewrite as $\frac{1}{g(y)}dy = f(x)dx$.
2. Integrate both sides:
$$
\int \frac{1}{g(y)}dy = \int f(x)dx + C
$$
3. Solve for y explicitly if possible.

Explicit Solution:
After integration, the solution takes the form:
$$
F(y) = G(x) + C
$$
where F and G are antiderivatives of the respective functions.

Solving Linear First-Order Equations

Equations of the form:
$$ \frac{dy}{dx} + P(x)y = Q(x) $$

Method: Integrating Factor

1. Identify the integrating factor:
$$
\mu(x) = e^{\int P(x) dx}
$$
2. Multiply the entire differential equation by $\mu(x)$:
$$
\frac{d}{dx} \left[\mu(x) y \right] = \mu(x) Q(x)
$$
3. Integrate both sides:
$$
\mu(x) y = \int \mu(x) Q(x) dx + C
$$
4. Solve for y:
$$
y(x) = \frac{1}{\mu(x)} \left(\int \mu(x) Q(x) dx + C \right)
$$

Explicit Solution:
Expressed explicitly as:
$$
y(x) = \frac{1}{e^{\int P(x) dx}} \left(\int e^{\int P(x) dx} Q(x) dx + C \right)
$$

Solving Exact Equations

An equation $M(x,y) + N(x,y) \frac{dy}{dx} = 0$ is exact if:
$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Method:
1. Find a potential function $\Psi(x,y)$ such that:
$$
\frac{\partial \Psi}{\partial x} = M(x,y), \quad \frac{\partial \Psi}{\partial y} = N(x,y)
$$
2. Integrate M with respect to x:
$$
\Psi(x,y) = \int M(x,y) dx + h(y)
$$
3. Differentiate Ψ with respect to y and set equal to $N(x,y)$ to find $h(y)$.
4. The general solution:
$$
\Psi(x,y) = C
$$

Explicitly Expressing the Solution as a Function of the Independent Variable

After solving the differential equation, the goal is to write the general solution explicitly as a function of the independent variable x, involving arbitrary constants that encapsulate the family of solutions.

General Solution for First-Order Equations

Depending on the method used, the explicit form may vary:

- Separable Equations:
$$
y = \text{Function of } x \text{ obtained from inversion of } F(y) = G(x) + C
$$
- Linear Equations:
$$
y(x) = \frac{1}{\mu(x)} \left(\int \mu(x) Q(x) dx + C \right)
$$
which is explicitly in terms of x.

- Exact Equations:
$$
\Psi(x,y) = C
$$
Solve for y explicitly if possible:
$$
y = \text{Function of } x \text{ and } C
$$

Explicit Solution for Higher-Order Equations

For linear higher-order equations with constant coefficients, the solution typically involves:

- Characteristic equations
- Roots of the characteristic polynomial
- General linear combination of exponential, sine, or cosine functions

Example:
For $y'' - 3 y' + 2 y = 0$,

Solve the characteristic equation:
$$
r^2 - 3r + 2 = 0
$$
which factors as:
$$
(r - 1)(r - 2) = 0
$$
so roots are $r=1$ and $r=2$. The general solution:
$$
y(x) = C_1 e^{x} + C_2 e^{2x}
$$

This explicit form clearly expresses the solution as a function of x with arbitrary constants C_1, C_2.

Examples Demonstrating Explicit Solutions

Example 1: Separable Differential Equation

Solve:
$$
\frac{dy}{dx} = y \left(1 - y \right)
$$

Solution:
1. Rewrite:
$$
\frac{dy}{y(1 - y)} = dx
$$
2. Partial fractions:
$$
\frac{1}{y(1 - y)} = \frac{1}{y} + \frac{1}{1 - y}
$$
3. Integrate:
$$
\int \left(\frac{1}{y} + \frac{1}{1 - y} \right) dy = \int dx
$$
$$
\ln |y| - \ln |1 - y| = x + C
$$
4. Simplify:
$$
\ln \left| \frac{y}{1 - y} \right| = x + C
$$
5. Exponentiate:
$$
\left| \frac{y}{1 - y} \right| = e^{x+C} = Ae^{x}
$$
where $A = e^{C} > 0$.

Explicit solution:
$$
\frac{y}{1 - y} = \pm Ae^{x}
$$
Solve for y:

$$
y = \frac{\pm Ae^{x}}{1 + \pm Ae^{x}}
$$
which explicitly expresses y as a function of x.

Example 2: Linear First-Order Equation

Solve:
$$
\frac{dy}{dx} + 2 y = e^{x}
$$

Solution:
1. Identify $P(x) = 2$, $Q(x) = e^{x}$.
2. Compute integrating factor:
$$
\mu(x) = e^{\int 2 dx} = e^{2x}
$$
3. Multiply the equation:
$$
e^{2x} \frac{dy}{dx} + 2 e^{2x} y = e^{3x}
$$
which simplifies to:
$$
\frac{d}{dx} \left(e^{2x} y \right) = e^{3
$$

Frequently Asked Questions

How do you find the general solution of a first-order linear differential equation?

To find the general solution of a first-order linear differential equation, write it in the form $dy/dx + P(x)y = Q(x)$, then find the integrating factor $\mu(x) = e^{\int P(x)dx}$. Multiply the entire equation by $\mu(x)$, rewrite as a derivative of $(\mu(x)y)$, integrate both sides, and solve for $y(x)$.

What is the method to find the general solution of a homogeneous differential equation?

For a homogeneous differential equation, express it in terms of y/x or other substitutions to reduce it to a separable form. Then, separate variables and integrate to find the explicit general solution as a function of x.

How can you express the general solution explicitly as a function of x for a second-order differential equation?

Solve the characteristic equation associated with the differential equation to find roots. Use these roots to write the general solution as a linear combination of exponential functions, providing an explicit expression of $y(x)$.

What are the steps to find the general solution of an exact differential equation?

Identify that the differential equation is exact, find the potential function $F(x,y)$ by integrating $M(x,y)$ with respect to x and N(x,y) with respect to y, ensuring consistency. The general solution is then given by F(x,y) = C, expressed explicitly as y = y(x) if possible.

How do you express the general solution explicitly when solving a differential equation using separation of variables?

Separate variables to get all y terms on one side and x terms on the other. Integrate both sides with respect to their variables, then solve the resulting equation for y explicitly as a function of x, including the constant of integration.

What is the significance of expressing the general solution explicitly as a function of x in differential equations?

Expressing the solution explicitly as a function of x allows for direct evaluation of y at any given x, facilitates interpretation of the solution's behavior, and makes it easier to apply initial or boundary conditions to find particular solutions.

Additional Resources

Find The General Solution Of The Following Equation. Express The Solution Explicitly As A Function Of is a fundamental instruction that guides students and mathematicians alike in solving differential equations. This directive emphasizes the importance of deriving comprehensive solutions that encompass all possible particular solutions, thereby capturing the entire set of solutions in a generalized form. Understanding how to approach such problems is essential for many fields, including physics, engineering, economics, and biological sciences, where modeling dynamic systems hinges on solving differential equations accurately and explicitly.

In this article, we will explore the process of finding the general solution of differential equations, emphasizing the importance of expressing solutions explicitly as functions of the independent variable(s). We will break down different types of equations, discuss various methods for solving them, and highlight best practices, common challenges, and tips to ensure clarity and correctness.

Understanding Differential Equations and Their Solutions

What Is a Differential Equation?

A differential equation is an equation that involves an unknown function and its derivatives. These equations describe how a quantity changes concerning one or more variables. For example, in physics, they might model motion or heat transfer; in biology, population dynamics; and in economics, growth models.

Differential equations are broadly classified into:

- Ordinary Differential Equations (ODEs): Involving derivatives with respect to a single independent variable.
- Partial Differential Equations (PDEs): Involving derivatives with respect to multiple variables.

This article primarily focuses on ordinary differential equations, given their foundational role and relative simplicity.

The Concept of a General Solution

The general solution of a differential equation encompasses all possible particular solutions. It typically contains arbitrary constants that arise during integration, representing the degrees of freedom inherent in the problem. For example, integrating a first-order differential equation yields a solution with one arbitrary constant; a second-order yields two, and so on.

Expressing the solution explicitly as a function of the independent variable(s) means solving the differential equation in the form:

$$y = f(x)$$

where y is expressed directly in terms of x (and parameters if any). Sometimes, solutions are implicit, but the goal is often to make them explicit for ease of interpretation and application.

Methods for Finding the General Solution

Different classes of differential equations require different solution techniques. Below, we explore the most common methods, their features, and when to use them.

1. Separating Variables

Applicable to: First-order differential equations where variables can be separated.

Method: Rewrite the equation so all y-terms are on one side and all x-terms on the other, then integrate both sides.

Example:

$$\frac{dy}{dx} = g(x)h(y)$$

Rearranged as:

$$\frac{1}{h(y)}dy = g(x)dx$$

Solution steps:

- Separate variables.
- Integrate both sides:

$$\int \frac{1}{h(y)}dy = \int g(x)dx + C$$

- Solve for y explicitly, if possible.

Features:

- Simple and straightforward.
- Effective for many first-order equations.
- Usually yields an explicit solution or an implicit relation.

Pros:

- Easy to implement.
- Provides a clear pathway to the general solution.

Cons:

- Not applicable if variables cannot be separated.
- Integration may be complicated or impossible analytically.

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2. Integrating Factors

Applicable to: First-order linear differential equations of the form:

$$ \frac{dy}{dx} + P(x)y = Q(x) $$

Method: Multiply through by an integrating factor $\mu(x) = e^{\int P(x) dx}$ to make the left side an exact derivative.

Solution steps:

- Find the integrating factor $\mu(x)$.
- Rewrite the equation:

$$ \frac{d}{dx}[\mu(x)y] = \mu(x)Q(x) $$

- Integrate both sides:

$$ \mu(x)y = \int \mu(x)Q(x) dx + C $$

- Solve for y:

$$ y = \frac{1}{\mu(x)} \left(\int \mu(x)Q(x) dx + C \right) $$

Features:

- Systematic and powerful for linear equations.
- Yields an explicit general solution with arbitrary constants.

Pros:

- Widely applicable for linear first-order equations.
- Provides a clear, structured solution process.

Cons:

- Not applicable to nonlinear equations.
- Integration may be complex.

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3. Homogeneous and Exact Equations

Homogeneous Equations:

- For equations where the RHS and LHS can be expressed as functions of y/x or similar ratios.
- Substitution $v = y/x$ often simplifies the problem.

Exact Equations:

- Equations that can be written as the total differential of a function:

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

- If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the equation is exact.

Method:

- Find a potential function $\Psi(x,y)$ such that:

$$\frac{\partial \Psi}{\partial x} = M(x,y), \quad \frac{\partial \Psi}{\partial y} = N(x,y)$$

- The general solution is $\Psi(x,y) = C$.

Features:

- Provides an explicit form when possible.
- Useful for first-order equations that satisfy the exactness condition.

Pros:

- Once identified as exact, straightforward to integrate.
- Yields implicit solutions that can sometimes be solved explicitly.

Cons:

- Not all equations are exact; may need to find an integrating factor.

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4. Higher-Order Differential Equations

Methods for second or higher-order equations include:

- Reduction of order: When one solution is known, find others.
- Characteristic equations: For linear constant coefficient equations.
- Variation of parameters: To find particular solutions.
- Undetermined coefficients: For equations with specific types of forcing functions.

Each method involves systematic steps to find the general solution and then express it explicitly as a function of the independent variable.

Expressing Solutions Explicitly as Functions of the Independent Variable

After solving the differential equation, the next step is to write the general solution explicitly as a function of x, which often involves algebraic manipulation or solving algebraic equations.

Implications of Explicit Solutions

- Facilitate direct evaluation at specific points.
- Simplify understanding the behavior of the solution.
- Enable straightforward application to initial or boundary conditions.

Challenges in Obtaining Explicit Solutions

- Some solutions are inherently implicit, involving transcendental functions.
- Algebraic manipulation to isolate \(y \) can be complex or impossible analytically.
- Numerical methods may be necessary where explicit solutions are not feasible.

Applying Initial Conditions and Finding Particular Solutions

The general solution contains arbitrary constants. To find a particular solution relevant to a specific problem, initial conditions or boundary conditions are used.

Procedure:

- Plug the given conditions into the general solution.
- Solve for the constants.
- Write the explicit particular solution.

Note: The process underscores the importance of expressing solutions explicitly, as it simplifies the application of initial/boundary conditions.

Summary of Best Practices and Tips

- Identify the type of differential equation first: This guides the choice of method.
- Attempt separation of variables early: If possible, this is often the simplest route.
- Check for linearity: Use integrating factors where applicable.
- Verify exactness: If an equation is suspected to be exact, verify and proceed accordingly.
- Use substitutions judiciously: Homogeneous equations and other special forms often benefit from substitution.
- Aim for explicit solutions: While some differential equations only yield implicit solutions, strive to manipulate results into explicit forms for clarity.
- Validate solutions: Differentiate to verify that the obtained solution satisfies the original differential equation.

Conclusion

Finding the general solution of the following equation, and expressing it explicitly as a function of the independent variable, is a cornerstone skill in differential equations. The process involves understanding the structure of the equation, selecting an appropriate solution method, and performing algebraic or calculus-based manipulations to arrive at a comprehensive, explicit form. While some equations lend themselves to straightforward solutions, others may require more advanced techniques or numerical methods. Mastery of these methods enhances one's ability to model, analyze, and interpret complex systems across scientific disciplines.

Whether dealing with simple separable equations or complex higher-order linear equations, the key is clarity in approach, thoroughness in solution derivation, and precision in expressing the solution explicitly. This approach ensures that solutions are not just mathematically sound but also practically applicable, facilitating deeper insights into the behavior of dynamic systems.

Features and Summary

- Versatile Methods: Separation of variables, integrating factors, substitutions, and more.
- Explicit Solutions: Critical for application and understanding.
- Challenges Addressed: Implicit solutions, complex integrations, non-separable equations.
- Practical Tips: Always verify solutions and consider initial/boundary conditions.

By mastering these techniques and

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