Find The Length Of The Following Two-dimensional Curve.r(t)=2cost,2sint, For0tThe Length Of The Curve

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Understanding how to determine the length of a parametric curve is fundamental in calculus and geometry. In this article, we explore the process of calculating the length of the specific two-dimensional curve defined by the parametric equations $r(t) = (2\cos t, 2\sin t)$ over the interval $0 \leq t \leq \pi/2$. By the end of this comprehensive guide, you will have a clear understanding of the methods involved, the mathematical principles underpinning the process, and practical applications of curve length calculations.

Introduction to Curve Length in Calculus

Before diving into the specific example, it's essential to grasp the general concept of the length of a parametric curve.

What Is a Parametric Curve?

A parametric curve in the plane is described by a pair of functions:

- $x = x(t)$
- $y = y(t)$

where t is a parameter that varies over an interval [a, b]. These functions provide the coordinates of points along the curve as t changes.

Why Calculate Curve Length?

Knowing the length of a curve is crucial in numerous fields such as physics, engineering, and computer graphics. It helps in understanding distances along paths, designing curves and shapes, and analyzing motion.

The Mathematical Foundation of Curve Length

The length of a parametric curve $r(t) = (x(t), y(t))$, for t in [a, b], is given by the line integral:

L = ? ? ? ? $[(dx/dt)^2 + (dy/dt)^2]\, dt$

This formula calculates the accumulated length by integrating the infinitesimal distances along the curve.

Deriving the Formula

Given the parametric equations:

$x(t)$ and $y(t)$

The derivatives are:

- dx/dt
- dy/dt

The differential arc length ds at each t is:

$ds = ⍰\ [(dx/dt)^2 + (dy/dt)^2]\ dt$

Summing these infinitesimal segments from $t = a$ to $t = b$ yields the total length.

Applying the Formula to $r(t) = (2\cos t, 2\sin t)$

Now, we focus on the specific curve:

$r(t) = (2\cos t, 2\sin t)$, where t ⍰ $[0, \pi/2]$

This curve traces a quarter of a circle with radius 2, starting at (2, 0) and ending at (0, 2).

Step 1: Compute the derivatives dx/dt and dy/dt

Given:

$x(t) = 2\cos t$

$y(t) = 2\sin t$

Derivatives are:

$dx/dt = -2\sin t$

$dy/dt = 2\cos t$

Step 2: Compute the integrand ⍰ $[(dx/dt)^2 + (dy/dt)^2]$

Calculating:

$(dx/dt)^2 + (dy/dt)^2 = (-2\sin t)^2 + (2\cos t)^2$

$= 4\sin^2 t + 4\cos^2 t$

Factor out 4:

$= 4 (\sin^2 t + \cos^2 t)$

Using the Pythagorean identity:

$\sin^2 t + \cos^2 t = 1$

Therefore:

$(dx/dt)^2 + (dy/dt)^2 = 4\ 1 = 4$

The square root is:

$\sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{4} = 2$

Step 3: Set up the integral for the length

The total length L over $t \in [0, \pi/2]$ is:

$L = \int_0^{\pi/2} 2 \, dt$

Since the integrand is constant, the integral simplifies to:

$L = 2 (\pi/2 - 0) = 2 (\pi/2) = \pi$

Interpretation of the Result

The length of the curve $r(t) = (2\cos t, 2\sin t)$, for t from 0 to $\pi/2$, is π units.

This makes sense geometrically because the curve traces a quarter circle of radius 2, and the circumference of a circle with radius r is $2\pi r$. Therefore, a quarter of this circle has length:

$(1/4)\, 2\pi r = (1/4)\, 2\pi\, 2 = \pi$

which confirms our calculation.

Generalization and Related Concepts

Understanding this specific example provides insight into more complex curve length calculations.

Curves of Different Shapes

- Ellipses, parabolas, and hyperbolas can be analyzed similarly, but often require more involved integrations.
- For circles, the formula simplifies due to symmetry and known circumference.

Parametric vs. Cartesian Methods

While parametric equations often simplify the process, sometimes converting the curve into Cartesian form helps, especially for straightforward shapes.

Arc Length in Higher Dimensions

The concept extends to three-dimensional curves, where the length involves derivatives of $x(t)$, $y(t)$, and $z(t)$:

$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} \, dt$

Practical Applications of Curve Length Calculations

Calculating the length of curves is not just a theoretical exercise; it has practical applications across various domains:

- **Engineering:** Designing roads, rails, and pipelines along specified paths.
- **Physics:** Analyzing the path length of particles moving along curved trajectories.
- **Computer Graphics:** Rendering curves and animations accurately.
- **Biology:** Measuring the length of biological structures like blood vessels or nerve fibers.
- **Mathematics Education:** Enhancing understanding of calculus concepts through practical examples.

Additional Examples and Practice Problems

To reinforce understanding, consider the following practice problems:

Example 1: Complete the Length of a Full Circle with Radius 2

Calculate the total circumference of the circle $r(t) = (2\cos t, 2\sin t)$ over $t \in [0, 2\pi]$.

Solution:

As shown earlier, the length of a full circle with radius 2 is:

$L = 2\pi r = 2\pi \, 2 = 4\pi$

Example 2: Length of a Semi-Circular Arc

Calculate the length of the semi-circle $r(t) = (2\cos t, 2\sin t)$, for $t \in [0, \pi]$.

Solution:

Since the full circle length is 4π, half of it (semi-circle) is:

$L = 2\pi$

Practice Problem: Elliptical Arc Length

Given an ellipse $x = a \cos t$, $y = b \sin t$, find the length of the quarter ellipse from $t=0$ to $t=\pi/2$.

Summary

Calculating the length of a parametric curve involves understanding the derivatives of the component functions and applying the arc length integral. For the specific case of r(t) = (2cos t, 2sin t), the process simplifies due to the constant magnitude of the derivative, leading to a straightforward integral that yields the quarter circle length of π units. Mastery of these techniques provides essential tools for analyzing curves in mathematics, physics, engineering, and beyond.

References

- Stewart, James. Calculus: Early Transcendentals. Cengage Learning, 8th Edition.
- Thomas, George B., and Ross L. Finney. Calculus and Analytic Geometry. Addison-Wesley.
- Online resources such as Khan Academy and Paul's Online Math Notes provide further explanations and practice problems related to curve length calculations.

Note: Always verify the interval over which you are integrating to ensure your calculation matches the segment of the curve you are interested in.

Frequently Asked Questions

What is the formula to find the length of a parametric curve r(t) = (x(t), y(t)) from t = a to t = b?

The length L of the curve is given by L = ⍰ ⍰ ⍰ ⍰ [(dx/dt)^2 + (dy/dt)^2] dt.

Given r(t) = (2 cos t, 2 sin t), how do we compute the derivatives dx/dt and dy/dt?

dx/dt = -2 sin t and dy/dt = 2 cos t.

What is the expression for the integrand ⍰ [(dx/dt)^2 + (dy/dt)^2] for the given r(t)?

The integrand is ⍰ [(-2 sin t)^2 + (2 cos t)^2] = ⍰ [4 sin^2 t + 4 cos^2 t] = ⍰ [4 (sin^2 t + cos^2 t)] = ⍰ 4 = 2.

What are the limits of integration for t in the problem?

t varies from 0 to π/2 as specified in the problem.

How do we compute the length of the curve r(t) = (2 cos t, 2 sin t) from t=0 to t=π/2?

Since the integrand is constant at 2, the length L = ⍰ $_0$^{π/2} 2 dt = 2 × (π/2 - 0) = π.

What is the geometric interpretation of the curve r(t) = (2 cos t, 2 sin t)?

It traces a circle of radius 2 centered at the origin.

What is the length of the entire circle traced by r(t) = (2 cos t, 2 sin t)

from t=0 to t=2π?

The circumference of the circle is 2π × radius = 2π × 2 = 4π.

How does the calculated length from t=0 to t=π/2 relate to the circle's total circumference?

The length from t=0 to t=π/2 is a quarter of the circle's circumference, which is (1/4) × 4π = π.

Can the formula used to find the curve length be applied to any parametric curve? If yes, what are the conditions?

Yes, the formula applies to any smooth parametric curve where derivatives dx/dt and dy/dt are continuous over the interval; the length is computed as $L = \int \sqrt{[(dx/dt)^2 + (dy/dt)^2]}\, dt$.

What is the final answer for the length of the curve r(t) = (2 cos t, 2 sin t) from t=0 to t=π/2?

The length of the curve over this interval is π.

Additional Resources

Find The Length Of The Following Two-dimensional Curve r(t) = (2cos t, 2sin t), for 0 ≤ t ≤ π

Understanding how to find the length of a curve is a fundamental skill in calculus, especially when dealing with parametric equations. The problem "Find The Length Of The Following Two-dimensional Curve r(t) = (2cos t, 2sin t), for 0 ≤ t ≤ π" provides a classic example involving a parametric representation of a circle segment. In this article, we will thoroughly explore the process, step-by-step, to derive the arc length of this curve, highlighting key concepts, formulas, and practical tips along the way.

Introduction to Curve Length in Calculus

Before diving into the specific problem, it's essential to understand the general idea behind calculating the length of a curve.

What is Curve Length?

The length of a curve in the plane between two points is the measure of the shortest path along the curve connecting those points. For a smooth curve described parametrically by functions $x(t)$ and $y(t)$, the arc length L from $t = a$ to $t = b$ is given by:

$$
L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} \, dt
$$

This formula stems from the Pythagorean theorem, considering the infinitesimal segments along the curve.

Step-by-Step Breakdown of the Problem

Now, let's analyze the specific parametric curve:

$$
r(t) = (x(t), y(t)) = (2\cos t, 2\sin t)
$$

for t in the interval $[0, \pi]$.

1. Recognize the Geometric Nature of the Curve

The parametric equations describe a circle:

- The x-coordinate: $2\cos t$
- The y-coordinate: $2\sin t$

As t varies from 0 to π, the point moves along the upper half of a circle with radius 2.

Key Insight: Since $x(t)^2 + y(t)^2 = (2\cos t)^2 + (2\sin t)^2 = 4(\cos^2 t + \sin^2 t) = 4$, the curve is a circle centered at the origin with radius 2.

2. Determine the Derivatives dx/dt and dy/dt

To compute the arc length, we need the derivatives:

$$
\frac{dx}{dt} = -2 \sin t
$$

$$
\frac{dy}{dt} = 2 \cos t
$$

3. Set Up the Arc Length Integral

Using the arc length formula:

$$
L = \int_0^{\pi} \sqrt{\left(-2 \sin t \right)^2 + \left(2 \cos t \right)^2} \, dt
$$

Simplify the integrand:

$$
L = \int_0^{\pi} \sqrt{4 \sin^2 t + 4 \cos^2 t} \, dt
$$

Factor out 4:

$$
L = \int_0^{\pi} \sqrt{4 (\sin^2 t + \cos^2 t)} \, dt
$$

Recall the Pythagorean identity:

$$
\sin^2 t + \cos^2 t = 1
$$

So,

$$
L = \int_0^{\pi} \sqrt{4 \times 1} \, dt = \int_0^{\pi} 2 \, dt
$$

\]

Calculating the Arc Length

The integral simplifies to:

\[
L = 2 \int_0^{\pi} dt
\]

which is straightforward:

\[
L = 2 \left[t \right]_0^{\pi} = 2 (\pi - 0) = 2\pi
\]

Result: The length of the curve \(r(t) = (2\cos t, 2\sin t) \) from \(t=0 \) to \(t=\pi \) is \(2\pi \).

Geometric Interpretation

The curve described is a semicircular arc:

- Radius: 2
- Angle swept: \(\pi \) radians (180 degrees)

The length of a circular arc with radius \(r \) over an angle \(\theta \) radians is:

\[
L = r \theta
\]

Applying this formula:

\[
L = 2 \times \pi = 2\pi
\]

which aligns perfectly with our integral calculation.

Additional Insights and Practical Tips

1. Recognizing Standard Curves

- The parametric equations \(x(t) = r \cos t \), \(y(t) = r \sin t \) describe circles.
- The length of the entire circle is \(2\pi r \).
- For a segment, multiply the radius by the angle in radians.

2. Choosing the Limits

- When working with parametric equations, always check the interval for \(t \) to interpret the segment's geometric nature.
- In this case, \(t \) from 0 to \(\pi \) corresponds to the upper semicircle.

3. Simplify the Integral

- Recognize trigonometric identities to simplify the integrand.
- When derivatives involve sine and cosine, the squared sum often simplifies to a constant, as seen here.

4. Alternative Methods

- Geometric approach: Using the circle formula directly.
- Numerical approximation: For more complex curves, numerical methods like Simpson's rule can approximate the length.

Summary of Key Steps

Step	Action	Explanation
1	Recognize the curve	It's a circle of radius 2.
2	Find derivatives	$dx/dt = -2 \sin t$, $dy/dt = 2 \cos t$.
3	Set up the integral	$L = \int_{0}^{\pi} \sqrt{(-2 \sin t)^2 + (2 \cos t)^2} dt$.
4	Simplify integrand	$\sqrt{4 \sin^2 t + 4 \cos^2 t} = 2$.
5	Compute integral	$L = 2 \int_0^{\pi} dt = 2 \pi$.

Final Thoughts

Calculating the length of a parametric curve involves identifying the geometric nature of the curve, computing derivatives, and setting up the correct integral. The elegant simplification in this problem reflects the beauty of symmetry and identities in calculus, making it straightforward to determine that the length of the upper semicircular arc of radius 2 from $t=0$ to $t=\pi$ is 2π.

Understanding these principles not only helps solve similar problems efficiently but also deepens your appreciation for the interconnectedness of calculus and geometry. Whether working with circles, ellipses, or more complex curves, mastering the arc length formula is an invaluable tool for any mathematician, engineer, or scientist.

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