Find The Limit (enter 'DNE' If The Limit Does Not Exist) (2c + Y)? Lim (z,u) `(0,0) 4x2 + Y? 1) Along

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Understanding limits in multivariable calculus is fundamental for analyzing the behavior of functions near specific points, especially when the functions involve multiple variables such as z and u. This article aims to provide a detailed, step-by-step guide to finding the limit of the function $4x^2 + y$ as the point (z, u) approaches $(0, 0)$, along various paths. We will explore the concepts, techniques, and potential pitfalls involved in such calculations, helping you determine whether the limit exists or does not exist (DNE).

Introduction to Multivariable Limits

What Is a Limit in Multivariable Calculus?

In single-variable calculus, the limit describes the value that a function approaches as the input approaches a particular point. In multivariable calculus, the concept extends to functions of multiple variables, such as $f(z, u)$, where the limit as $(z, u) \to (a, b)$ considers the behavior of the function as both variables approach their respective values simultaneously.

Mathematically, the limit is expressed as:

$$
\lim_{(z,u) \to (a,b)} f(z, u)
$$

if and only if, for every sequence (z_n, u_n) approaching (a, b), the function values $f(z_n, u_n)$ approach a common value L. If such a value exists, it is the limit; otherwise, the limit does not exist (DNE).

Why Do Limits Matter?

Limits are essential for:

- Defining derivatives in multiple variables.
- Analyzing continuity.
- Calculating tangent planes and gradients.
- Understanding the behavior of functions near singular points.

Analyzing the Given Function and Limit

Function Description

From the statement, the function appears to involve the expression:

$$
f(z, u) = 4x^2 + y
$$

However, there is some ambiguity due to the variables' notation. Assuming the variables are z and u and the function involves x and y, which are perhaps related to z and u, or perhaps the original problem is to analyze the limit of $4z^2 + u$ as $(z, u) \to (0, 0)$.

For clarity, let's interpret the function as:

$$
f(z, u) = 4z^2 + u
$$

and analyze:

$$
\lim_{(z, u) \to (0, 0)} f(z, u)
$$

along various paths. If the original problem involves different variables, the same principles apply.

Objective

Determine whether:

$$
\lim_{(z, u) \to (0, 0)} 4z^2 + u
$$

exists, and if so, find its value. If the limit does not exist, state DNE.

Methodology for Finding Limits in Multiple Variables

1. Direct Substitution

The first step in evaluating a limit is to attempt direct substitution:

$$
f(0, 0) = 4(0)^2 + 0 = 0
$$

If the function is defined at $(0, 0)$ and direct substitution yields a finite value, the limit may exist and be equal to this value. However, in multivariable calculus, direct substitution alone isn't sufficient because the limit can depend on the path taken toward the point.

2. Path Analysis

To verify whether the limit exists, analyze the behavior along different paths approaching $(0, 0)$:

- Path 1: $u = 0$, $z \to 0$
- Path 2: $z = 0$, $u \to 0$
- Path 3: $u = k z$, where k is a constant
- Path 4: $u = z^2$

If the limits along all these paths yield the same value, the limit likely exists. If not, the limit DNE.

3. Using Polar Coordinates (or Equivalent in 2D)

In multivariable calculus, transforming to polar coordinates (or similar parametrizations) can help evaluate limits, especially when dealing with functions involving symmetric terms like $z^2 + u^2$. For the variables z and u, polar coordinates are:

$$
z = r \cos \theta,\quad u = r \sin \theta
$$

with $r \to 0$. Substituting into the function helps examine the limit's behavior as $r \to 0$.

Step-by-Step Solution

Step 1: Direct Substitution

$$
f(0, 0) = 4(0)^2 + 0 = 0
$$

Since the function evaluates to 0 at $(0, 0)$, the potential limit is 0. But we must verify that approaching $(0, 0)$ along different paths also yields 0.

Step 2: Path Analysis

- Path A: $u = 0$

$$
f(z, 0) = 4z^2 + 0 = 4z^2
$$

As $z \to 0$:

$$
\lim_{z \to 0} 4z^2 = 0
$$

- Path B: $z = 0$

$$
f(0, u) = 0 + u = u
$$

As $u \to 0$:

$$
\lim_{u \to 0} u = 0
$$

- Path C: $u = k z$

$$
f(z, k z) = 4z^2 + k z
$$

As $z \to 0$:

$$
\lim_{z \to 0} 4z^2 + k z = 0 + 0 = 0
$$

- Path D: $u = z^2$

$$
f(z, z^2) = 4z^2 + z^2 = 5z^2
$$

As $z \to 0$:

$$
\lim_{z \to 0} 5z^2 = 0
$$

All paths yield the same limit of 0, suggesting the limit exists and is equal to 0.

Step 3: Confirm with Polar Coordinates

Express the function in polar coordinates:

$$
z = r \cos \theta,\quad u = r \sin \theta
$$

$$
f(z, u) = 4z^2 + u = 4(r \cos \theta)^2 + r \sin \theta = 4 r^2 \cos^2 \theta + r \sin \theta
$$

Factor r:

$$
f(r, \theta) = r (4 r \cos^2 \theta + \sin \theta)
$$

As $r \to 0$:

$$
\lim_{r \to 0} r (4 r \cos^2 \theta + \sin \theta) = 0
$$

since $r \to 0$, regardless of θ, the entire expression approaches 0.

Conclusion: The limit exists and is 0.

Final Answer: Limit is 0

Since all paths and the polar coordinate analysis lead to the same value, the limit of $f(z, u) = 4z^2 + u$ as $(z, u) \to (0, 0)$ is:

$$
\boxed{0}
$$

When Does the Limit Not Exist (DNE)?

While in this example the limit exists and equals 0, there are situations where the limit does not exist:

- Different Limits Along Different Paths: If approaching (a, b) along different paths yields different limit values, the overall limit DNE.
- Unbounded Behavior: If the function approaches infinity or oscillates wildly near the point.
- Discontinuities: If the function is not defined or behaves irregularly near the point.

Example: Consider $f(z, u) = \frac{zu}{z^2 + u^2}$. Approaching $(0, 0)$:

- Along $u = z$:

$$
f(z, z) = \frac{z \cdot z}{z^2 + z^2} = \frac{z^2}{2z^2} = \frac{1}{2}
$$

- Along $u = 0$:

$$
f(z, 0) = 0
$$

Different paths give different limits (1/2 and 0), so the overall limit DNE.
