Find The Maxima And Minima, And Where They Are Reached, Of The Function In F(x,y) = X + Y + Xy{(x,y):

Find The Maxima And Minima, And Where They Are Reached, Of The Function In F(x,y) = X + Y + Xy{(x,y):

Introduction

Understanding the maxima and minima of a function is fundamental in calculus and optimization. These points tell us where a function reaches its highest or lowest values within a certain domain, which is crucial in various fields such as economics, engineering, and physical sciences. In this article, we analyze the function $F(x, y) = x + y + xy$, aiming to determine its critical points, classify these points as maxima, minima, or saddle points, and identify where they are reached.

Understanding the Function $F(x, y) = x + y + xy$

Function Overview

The given function is a bivariate quadratic-like function:

$$
F(x, y) = x + y + xy
$$

It combines linear terms x, y with a bilinear term xy. Since it involves products of variables, its behavior can be more complex than simple linear functions, and it may exhibit saddle points, maxima, or minima depending on the domain.

Domain of the Function

The domain is typically all real pairs \((x, y)\), i.e., \(\mathbb{R}^2\). However, specific problems may restrict the domain, such as limiting the variables to a bounded region. For the sake of this analysis, we consider the entire plane unless specified otherwise.

Finding Critical Points

Critical points occur where the gradient of the function is zero. The gradient vector \(\nabla F(x, y)\) comprises the partial derivatives with respect to \(x\) and \(y\).

Calculating Partial Derivatives

\[
\frac{\partial F}{\partial x} = 1 + y
\]
\[
\frac{\partial F}{\partial y} = 1 + x
\]

Setting Partial Derivatives to Zero

To find critical points:

\[
\frac{\partial F}{\partial x} = 0 \Rightarrow 1 + y = 0 \Rightarrow y = -1
\]
\[
\frac{\partial F}{\partial y} = 0 \Rightarrow 1 + x = 0 \Rightarrow x = -1
\]

Thus, the only critical point in \(\mathbb{R}^2\) is at:

\[
\boxed{(x, y) = (-1, -1)}

\]

Classifying Critical Points

To classify the critical point, we analyze the second derivatives via the Hessian matrix.

Hessian Matrix

\[
H = \begin{bmatrix}
\frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\
\frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2}
\end{bmatrix}
\]

Calculating the second derivatives:

\[
\frac{\partial^2 F}{\partial x^2} = 0
\]
\[
\frac{\partial^2 F}{\partial y^2} = 0
\]
\[
\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = 1
\]

Therefore,

\[
H = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

Eigenvalues of the Hessian

Eigenvalues λ satisfy:

$$
\det(H - \lambda I) = 0
$$

$$
\det \begin{bmatrix}
-\lambda & 1 \\
1 & -\lambda
\end{bmatrix} = \lambda^2 - 1 = 0
$$

$$
\Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1
$$

Since the eigenvalues are of opposite signs, the Hessian is indefinite, indicating that the critical point at $(-1, -1)$ is a saddle point.

Summary of Critical Point Analysis

- Critical point at $(-1, -1)$
- Hessian matrix has eigenvalues ± 1
- The critical point is a saddle point, not a maximum or minimum

---

Extrema on Bounded Domains

While the critical point analysis indicates no maxima or minima in the entire plane, optimization often occurs within bounded regions. We explore how to find local maxima and minima within such regions.

Example: Domain $D = \{(x, y): a \leq x \leq b, c \leq y \leq d \}$

To find extrema within this bounded domain:

1. Evaluate $F(x, y)$ at the critical points inside D, if any.
2. Evaluate F at the boundary, i.e., along the edges $x = a, x = b, y = c, y = d$.
3. Check the corners of the domain.

Since the only critical point is at $(-1, -1)$, it is inside the domain D if and only if -1 lies within the bounds for x and y. The function's behavior along the boundaries can be analyzed by fixing one variable and examining the other.

Behavior of $F(x, y)$ at Boundaries and Corners

Along lines $x = \text{constant}$

$$
F(\text{constant}, y) = \text{constant} + y + (\text{constant}) \times y = \text{constant} + y + \text{constant} \times y
$$

$$
= \text{constant} + y (1 + \text{constant})
$$

The maximum or minimum occurs at boundary points for y, depending on the sign of $1 + \text{constant}$.

Similarly, along $y = \text{constant}$:

\[

$$
F(x, \text{constant}) = x + \text{constant} + x \times \text{constant} = \text{constant} + x (1 + \text{constant})
$$

Conclusion: Maxima and Minima of $F(x, y)$

- The only critical point in $\mathbb{R}^2$ is at $(-1, -1)$, which is a saddle point, not a maximum or minimum globally.
- The function is unbounded in certain directions, so it does not have a global maximum or minimum over the entire plane.
- To find local extrema, especially within a restricted domain, evaluate the function at critical points within the domain and along the boundary.
- The function's behavior at boundary points depends on the specific limits and the domain's shape.

Visual Interpretation and Graphical Behavior

Visualizing $F(x, y) = x + y + xy$ reveals a saddle-shaped surface with a saddle point at $(-1, -1)$. The function increases or decreases infinitely along specific directions, confirming the absence of global maxima or minima in the unbounded case.

Practical Implications

In optimization problems, recognizing that the critical point is a saddle point guides the decision-making process. For example:

- If the goal is to maximize F within a bounded region, evaluate F at the boundary and corners.
- If the goal is to understand the function's behavior, note that it tends toward infinity in certain directions, indicating no global extremum exists without domain restrictions.

Summary

- The function $F(x, y) = x + y + xy$ has a single critical point at $(-1, -1)$.
- This critical point is a saddle point, not a maximum or minimum.
- No global maxima or minima exist over $\mathbb{R}^2$ because the function is unbounded.
- For bounded regions, extrema are found at boundary points and corners, with the critical point serving as a saddle point within the interior.
- Analyzing the Hessian matrix confirms the saddle nature of the critical point.

References

- Stewart, J. (2015). Calculus: Early Transcendentals. Cengage Learning.
- Anton, H., Bivens, I., & Davis, S. (2012). Calculus: Multivariable. Wiley.
- Thomas, G. B., & Finney, R. L. (2002). Calculus and Analytic Geometry. Pearson.

This comprehensive analysis provides a clear understanding of the critical points, their classification, and the behavior of the function $F(x, y) = x + y + xy$, enabling effective optimization within various contexts.

Frequently Asked Questions

How do I determine the critical points of the function F(x, y) = x + y + xy?

To find critical points, compute the partial derivatives F_x and F_y, set them equal to zero, and solve the resulting system: $F_x = 1 + y = 0$ and $F_y = 1 + x = 0$. This yields the critical point at $(x, y) = (-1, -1)$.

How can I classify the critical point at (-1, -1) as a maximum, minimum, or saddle point?

Calculate the second derivatives $F_{xx} = 0$, $F_{yy} = 0$, and $F_{xy} = 1$. The discriminant $D = F_{xx} F_{yy} - (F_{xy})^2 = 0 0 - 1^2 = -1 < 0$, indicating that $(-1, -1)$ is a saddle point, not a maxima or minima.

Are there any boundary conditions or constraints for the function to consider when finding maxima and minima?

Yes, if the domain is constrained (e.g., x and y within certain regions), you need to evaluate the function along the boundaries and corners of that region to identify possible maxima and minima, in addition to critical points.

Where are the maxima and minima of $F(x, y) = x + y + xy$ located within an unconstrained domain?

Within an unconstrained domain, the critical point at (-1, -1) is a saddle point, and the function has no finite global maximum or minimum. The function can tend to infinity or negative infinity depending on the direction.

How does the function behave at infinity, and what does that imply about its maxima and minima?

As x or y tend to infinity or negative infinity, the function $F(x, y) = x + y + xy$ tends to infinity or negative infinity, indicating there are no finite global maxima or minima in the unconstrained case.

Can the method of Lagrange multipliers be used to find extrema under additional constraints for this function?

Yes, if the function is subject to constraints such as $g(x, y) = 0$, Lagrange multipliers can be applied to find potential maxima and minima on the constrained surface by solving the system of equations derived from the gradients.

What is the significance of the saddle point at (-1, -1) for the function's optimization?

The saddle point at (-1, -1) indicates a point where the function does not reach a maximum or minimum but rather changes curvature. It is critical for understanding the function's topology and for locating true extrema elsewhere if they exist under constraints.

Additional Resources

Find The Maxima And Minima, And Where They Are Reached, Of The Function In F(x,y) = X + Y + XY

Understanding how to identify the maxima and minima of a multivariable function is fundamental in calculus, optimization, and various applied sciences. In particular, analyzing the function $F(x,y) = x + y + xy$ offers valuable insights into the behavior of the function across its domain, highlighting the points where it attains its highest and lowest values (absolute maxima and minima) as well as the nature of these critical points. This article provides a comprehensive exploration of how to find these points, determine their locations, and interpret their significance.

Introduction to Maxima and Minima in Multivariable Functions

In the context of functions of two variables, maxima and minima refer to points in the domain where the function attains local or absolute highest or lowest values. Unlike single-variable calculus, where derivatives reduce to simple slopes, multivariable calculus involves analyzing partial derivatives and the behavior of the function in multiple directions.

Key concepts include:
- Critical points: Points where the gradient vector (composed of partial derivatives) is zero or undefined.
- Second derivative test: Used to classify critical points into maxima, minima, or saddle points.
- Boundary analysis: Since the domain might be restricted, maxima and minima could occur on the boundary as well.

Understanding these concepts is essential for analyzing $F(x,y) = x + y + xy$.

Analyzing the Function $F(x,y) = x + y + xy$

The function $F(x,y) = x + y + xy$ is a bilinear function that combines linear and multiplicative terms, making it interesting for extremum analysis.

Features of the function:
- It's continuous and differentiable everywhere in $\mathbb{R}^2$.
- The function exhibits saddle points and potential maxima or minima depending on the domain constraints.
- Its behavior is symmetric with respect to the variables, which can simplify analysis.

Step 1: Find the Critical Points

The first step in locating maxima and minima is to find the critical points by computing partial derivatives and setting them equal to zero.

Partial derivatives:

$$
\frac{\partial F}{\partial x} = 1 + y
$$

$$
\frac{\partial F}{\partial y} = 1 + x
$$

Setting derivatives to zero:

$$
1 + y = 0 \Rightarrow y = -1
$$

$$
1 + x = 0 \Rightarrow x = -1
$$

Critical point:

$$
(x, y) = (-1, -1)
$$

This is the only critical point in $\mathbb{R}^2$.

Step 2: Classify the Critical Point

To classify whether this critical point is a maximum, minimum, or saddle point, we examine the second derivatives and calculate the Hessian determinant.

Second derivatives:

$$
F_{xx} = 0,\quad F_{yy} = 0,\quad F_{xy} = 1
$$

Hessian matrix:

$$
H = \begin{bmatrix}
F_{xx} & F_{xy} \\
F_{yx} & F_{yy}
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
$$

Determinant of Hessian:

$$
D = F_{xx} \times F_{yy} - (F_{xy})^2 = (0)(0) - (1)^2 = -1
$$

Since $D < 0$, the critical point at $(-1, -1)$ is a saddle point. This indicates that the function does not have a local maximum or minimum at this point; instead, it changes from increasing in one direction to decreasing in another.

Maxima and Minima in the Unrestricted Domain

In the entire $\mathbb{R}^2$, the function $F(x,y) = x + y + xy$ does not possess a global maximum or minimum because the function's value tends to infinity or minus infinity as x and y grow large in certain directions.

Behavior at infinity:
- As $x, y \to +\infty$, $F(x,y) \to +\infty$.
- As $x, y \to -\infty$, $F(x,y) \to +\infty$.
- Along the line $y = -1$, $F(x, -1) = x -1 - x = -1$, a constant.

Thus, the function is unbounded above, and no global maximum exists over the entire plane.

Finding Maxima and Minima over Restricted Domains

Since the function is unbounded over $\mathbb{R}^2$, the analysis shifts to specific domains, such as bounded regions or domains with constraints. For example, consider a rectangular domain or a domain where x and y are limited within certain bounds.

Example domain:

$$
\Omega = \{ (x,y) \mid a \leq x \leq b,\ c \leq y \leq d \}
$$

Within such a domain, the maxima and minima occur either at critical points inside or on the boundary.

Step 3: Boundary Analysis

To find the absolute extrema on a bounded domain, evaluate F at:
- Critical points within the domain.
- The boundary lines $x = a, x = b, y = c, y = d$.
- The corners of the domain.

Procedure:
1. Compute F at the interior critical points (already identified: $(-1, -1)$ — check if within the domain).
2. Evaluate F along each boundary line:
- For $x = a$, $F(a, y) = a + y + a y$.
- For $x = b$, $F(b, y) = b + y + b y$.
- For $y = c$, $F(x, c) = x + c + x c$.
- For $y = d$, $F(x, d) = x + d + x d$.

3. Find the extrema along these boundary lines by treating them as single-variable functions in y or x and analyzing their maxima and minima.

4. Check the values at the four corners:

$$
(a, c),\quad (a, d),\quad (b, c),\quad (b, d)
$$

Practical Examples and Applications

Suppose we want to find the maximum and minimum of $F(x, y)$ on the square domain \(0 \leq x \leq

2, $0 \leq y \leq 2$.

Step-by-step:
- Critical points: $(-1, -1)$, which is outside the domain, so ignore.
- Boundary evaluations:
- $x=0$: $F(0, y) = 0 + y + 0 \times y = y$. Max at $y=2$, $F=2$, Min at $y=0$, $F=0$.
- $x=2$: $F(2, y) = 2 + y + 2 y = 2 + y + 2 y = 2 + 3 y$. Max at $y=2$, $F=2 + 6=8$; Min at $y=0$, $F=2$.
- $y=0$: $F(x,0) = x + 0 + x \times 0 = x$. Max at $x=2$, $F=2$; Min at $x=0$, $F=0$.
- $y=2$: $F(x,2) = x + 2 + 2 x = x + 2 + 2 x = 2 + 3 x$. Max at $x=2$, $F=2 + 6=8$; Min at $x=0$, $F=2$.

- Corners:
- $(0,0)$: $0+0+0=0$.
- $(0,2)$: $0+2+0=2$.
- $(2,0)$: $2+0+0=2$.
- $(2,2)$: $2+2+4=8$.

Results:
- Maximum value: 8 at $(2,2)$.
- Minimum value: 0 at $(0,0)$.

This demonstrates how boundary analysis combined with interior critical points helps identify extrema within a specified domain.

Features, Pros, and Cons of the Analysis Method

Features:
- Relies on critical point analysis via partial derivatives

Find The Maxima And Minima And Where They Are Reached Of The Function In F X Y X Y Xy X Y

Find The Maxima And Minima And Where They Are Reached Of The Function In F X Y X Y Xy X Y

Related Articles

- Edwardo Want To Buy A Kateboard That Cot $139. If The Ale Tax On Edwardo' Tate I 7%, How Much Will He
- Express The Integral As A Limit Of Riemann Sums Using Right Endpoints. Do Not Evaluate The Limit. 5 +
- Following The February Bank Reconciliation, The Accountant Made The Following Entry In The Journal Of

Back to Home