Find The Maximum And Minimum Values Of F(x, Y, Z)=xy+z On The Sphere X + Y=2 Points At Which They Are

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Introduction

In the realm of multivariable calculus, optimization problems constrained by geometric surfaces are both fascinating and challenging. One such problem involves finding the maximum and minimum values of a given function subject to a constraint. Specifically, consider the function $F(x, y, z) = xy + z$ constrained to the surface defined by the sphere $x + y = 2$. The goal is to determine the extremal values of F on this surface and identify the points where these extremal values occur.

Understanding how to approach such problems combines knowledge of Lagrange multipliers, surface equations, and substitution techniques. This article provides a comprehensive guide to solving this particular problem, illustrating the step-by-step process and underlying concepts fundamental to optimization in multivariable calculus.

Understanding the Problem

The Function and the Constraint

- Function to optimize: $F(x, y, z) = xy + z$
- Constraint surface: $x + y = 2$

The problem asks us to find the maximum and minimum possible values of F when (x, y, z) lies on the surface defined by $x + y = 2$. Since z appears linearly and independently in the function, the key is to understand how the constraint influences the variables x and y, and how z can be expressed or bounded.

Approach to the Problem

To find the extremal values, we typically employ the method of Lagrange multipliers or substitution, depending on the nature of the constraint and the function.

Step 1: Express z in terms of x and y

Since z appears linearly and separately, the maximum or minimum of F depends on the choice of x, y satisfying the constraint, and the value of z can be any real number unless additional constraints are specified.

However, if the problem assumes (x, y, z) lies on some specified surface (e.g., a sphere), the surface constrains z as well, which is crucial for solving the problem.

Step 2: Clarify the surface constraint

The problem states "on the sphere $x + y = 2$". Typically, a sphere is given by an equation like $x^2 + y^2 + z^2 = r^2$.

Given the description, the "sphere" appears to be incorrectly specified as $x + y = 2$, which is a plane, not a sphere.

Assumption: Likely, the problem intends to impose the constraint:

- $x + y = 2$ (a plane)

- And perhaps, the sphere is defined as $x^2 + y^2 + z^2 = R^2$

If the original problem is exactly as stated, then the constraint is just the plane $x + y = 2$.

Alternatively, if the problem involves the intersection of a sphere with the plane, then the surface is the intersection of:

- Sphere: $x^2 + y^2 + z^2 = R^2$

- Plane: $x + y = 2$

For clarity and completeness, we will consider the scenario where the function is optimized over the intersection of the sphere:

$$
x^2 + y^2 + z^2 = R^2
$$

and the plane:

\[
x + y = 2
\]

This is a common and meaningful interpretation of the problem.

Formulating the Problem with the Sphere and Plane

Assuming the sphere is:

\[
x^2 + y^2 + z^2 = R^2
\]

and the plane:

\[
x + y = 2
\]

Our goal is to find the maximum and minimum of:

\[
F(x, y, z) = xy + z
\]

subject to:

\[
\begin{cases}
x + y = 2 \\
x^2 + y^2 + z^2 = R^2
\end{cases}
\]

Step-by-Step Solution

Step 1: Express \(y \) in terms of \(x \)

From the plane:

$$
y = 2 - x
$$

Step 2: Express z in terms of x

Substitute $y = 2 - x$ into the sphere equation:

$$
x^2 + (2 - x)^2 + z^2 = R^2
$$

Simplify:

$$
x^2 + (4 - 4x + x^2) + z^2 = R^2
$$
$$
x^2 + 4 - 4x + x^2 + z^2 = R^2
$$
$$
2x^2 - 4x + 4 + z^2 = R^2
$$

Solve for z:

$$
z^2 = R^2 - 2x^2 + 4x - 4
$$

Since z appears as $\pm \sqrt{\dots}$, the maximum and minimum of F depend on the possible values of z given the domain of x.

Step 3: Express $F(x)$ as a function of x

Recall:

$$
F(x, y, z) = xy + z
$$

Substitute \(y = 2 - x \):

\[
F(x, z) = x(2 - x) + z = 2x - x^2 + z
\]

The extremal values of \(F \) occur at the maximum and minimum of:

\[
F(x, z) = 2x - x^2 + z
\]

where

\[
z = \pm \sqrt{R^2 - 2x^2 + 4x - 4}
\]

subject to the radicand being non-negative:

\[
R^2 - 2x^2 + 4x - 4 \geq 0
\]

Finding the Domain of \(x \)

Set:

\[
D(x) = R^2 - 2x^2 + 4x - 4 \geq 0
\]

This is a quadratic inequality in \(x \):

\[
-2x^2 + 4x + (R^2 - 4) \geq 0
\]

Divide through by \(-2\):

\[

x^2 - 2x - \frac{R^2 - 4}{2} \leq 0
\]

Let:

\[
A = 1, \quad B = -2, \quad C = - \frac{R^2 - 4}{2}
\]

The roots of the quadratic:

\[
x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}
\]

Calculate discriminant:

\[
\Delta = B^2 - 4AC = 4 - 4 \times 1 \times \left(- \frac{R^2 - 4}{2}\right) = 4 + 2(R^2 - 4)
\]

Simplify:

\[
\Delta = 4 + 2R^2 - 8 = 2R^2 - 4
\]

Note: For \(\Delta \geq 0 \), we require:

\[
2R^2 - 4 \geq 0 \Rightarrow R^2 \geq 2
\]

Assuming this condition holds, the roots are:

\[
x = \frac{2 \pm \sqrt{2R^2 - 4}}{2} = 1 \pm \frac{\sqrt{2R^2 - 4}}{2}
\]

Because the quadratic in \(x \) opens upwards (coefficient of \(x^2 \) is positive), the inequality:

\[
x^2 - 2x - \frac{R^2 - 4}{2} \leq 0

\]

holds between the roots:

$$
x \in \left[\, 1 - \frac{\sqrt{2R^2 - 4}}{2}, \quad 1 + \frac{\sqrt{2R^2 - 4}}{2} \,\right]
$$

Determining the Maximum and Minimum of $F(x)$

Recall:

$$
F(x) = 2x - x^2 + z
$$

with

$$
z = \pm \sqrt{R^2 - 2x^2 + 4x - 4}
$$

To maximize or minimize F, consider the two cases:

- When z takes the maximum value (positive square root)
- When z takes the minimum value (negative square root)

Hence, define:

$$
F_{\pm}(x) = 2x - x^2 \pm \sqrt{R^2 - 2x^2 + 4x - 4}
$$

The overall extremal values occur at

Frequently Asked Questions

How do I determine the maximum and minimum values of the function **$f(x, y, z) = xy + z$ subject to the constraint $x + y = 2$?**

You can use the method of Lagrange multipliers, setting up equations for the function and the constraint, then solving for critical points to find the maximum and minimum values.

What are the steps to find the maximum and minimum of $f(x, y, z) = xy + z$ on the sphere defined by $x + y = 2$?

First, express z in terms of x and y if necessary, apply the constraint $x + y = 2$, then set up the Lagrangian with a multiplier for the constraint, solve the resulting system to identify critical points, and evaluate f at those points.

Can I use substitution to simplify finding the extrema of $f(x, y, z) = xy + z$ on the plane $x + y = 2$?

Yes, by expressing y as $y = 2 - x$, you can reduce the problem to functions of x and z, then analyze the resulting function to find maximum and minimum values.

What points on the line $x + y = 2$ give the maximum and minimum of $f(x, y, z) = xy + z$?

These points depend on the value of z. For a fixed z, you can find the maximum and minimum of $xy = x(2 - x) = 2x - x^2$, which occurs at $x = 1$ for the maximum and at the endpoints for the minimum, then evaluate z accordingly.

Does the value of z affect the maximum and minimum of $f(x, y, z) = xy + z$ when constrained to $x + y = 2$?

Yes, since z appears as an additive term, the extrema of xy are independent of z; however, adding z shifts the overall maximum and minimum values vertically.

What are the explicit coordinates where the maximum and minimum of $f(x, y, z) = xy + z$ occur given the constraint $x + y = 2$?

The maximum occurs at $(x, y) = (1, 1)$ with any z value, as xy is maximized at $x = y = 1$. The minimum occurs at the endpoints $x = 0$ or $x = 2$, $y = 2$ or $y = 0$, with the corresponding z value, depending on the context.

How do I interpret the points at which the maximum and minimum of the function occur on the constrained surface?

They correspond to the critical points found via Lagrange multipliers or substitution, representing the locations on the surface $x + y = 2$ where the function reaches its highest and lowest values, often at boundary points or points of symmetry.

Additional Resources

Optimization on a Sphere: Finding the Maximum and Minimum of $f(x, y, z) = xy + z$ Subject to $x + y = 2$

Introduction

In the realm of multivariable calculus and constrained optimization, problems involving finding maximum and minimum values of functions under specific constraints are both fundamental and intriguing. They often model real-world scenarios—from engineering to economics—where optimal solutions are sought within given boundaries. Today, we explore such a problem: determining the maximum and minimum values of the function $f(x, y, z) = xy + z$ constrained to the surface of a sphere, with the additional restriction that $x + y = 2$.

This exploration not only demonstrates the elegance of calculus techniques but also highlights how geometric insights and algebraic methods combine to solve complex problems efficiently. Whether you're a student aiming to deepen your understanding or a professional seeking a systematic approach, this article provides an exhaustive breakdown of the process, including methods, reasoning, and key insights.

Setting the Stage: The Problem Statement

Objective: Find the maximum and minimum of the function

$$
f(x, y, z) = xy + z
$$

Subject to:

1. The constraint of a sphere: $x^2 + y^2 + z^2 = R^2$ (assuming a sphere of radius R)
2. The linear relation: $x + y = 2$

Why Is This Problem Interesting?

This problem exemplifies the classic constrained optimization scenario, where the goal is to optimize a function within a restricted domain. The constraints are:

- An algebraic surface: the sphere, representing a closed, bounded region in $\mathbb{R}^3$.
- A linear equation: $x + y = 2$, which effectively restricts the domain to a certain plane intersecting the sphere.

The challenge lies in combining these constraints to locate the points where the function reaches its extremal values, which frequently involves techniques like Lagrange multipliers, substitution, and geometric interpretations.

Approach Overview

To comprehensively analyze this problem, we'll proceed through the following steps:

1. Understanding the constraints: Interpreting the sphere and the linear relation.
2. Reducing the problem: Simplifying the constraints to express z in terms of x, y, or vice versa.
3. Applying the method of Lagrange multipliers: Handling the constraints systematically.
4. Finding the candidate points: Solving the resulting equations.
5. Evaluating the function at candidate points: Determining maximum and minimum values.
6. Verifying the points: Ensuring they satisfy all constraints.

Step 1: Understanding the Constraints

The Sphere $x^2 + y^2 + z^2 = R^2$

This is a standard surface in $\mathbb{R}^3$, symmetric and bounded. For simplicity, unless specified otherwise, we can consider R as a positive constant.

The Linear Constraint $x + y = 2$

This equation defines a plane in $\mathbb{R}^3$. The intersection of this plane with the sphere results in a circle (or empty set if the plane does not intersect the sphere).

Step 2: Reducing the Problem

The key to simplifying the problem is to eliminate variables where possible.

Given $x + y = 2$, we can express y in terms of x:

$$
y = 2 - x
$$

Now, the function becomes:

$$
f(x, y, z) = xy + z = x(2 - x) + z
$$

which simplifies to:

$$
f(x, z) = 2x - x^2 + z
$$

But z remains to be expressed using the sphere's equation.

Substituting $y = 2 - x$ into the sphere's equation:

$$
x^2 + y^2 + z^2 = R^2
$$

becomes:

$$
x^2 + (2 - x)^2 + z^2 = R^2
$$

Expanding:

$$
x^2 + (4 - 4x + x^2) + z^2 = R^2
$$

which simplifies to:

$$
2x^2 - 4x + 4 + z^2 = R^2
$$

Rearranged as:

$$
z^2 = R^2 - 2x^2 + 4x - 4
$$

The expression under the square root must be non-negative for real z:

$$
R^2 - 2x^2 + 4x - 4 \geq 0
$$

Step 3: Applying the Method of Lagrange Multipliers

Alternatively, for a thorough and systematic approach, we can use Lagrange multipliers to handle the constraints directly.

Define the Lagrangian:

$$
\mathcal{L}(x, y, z, \lambda, \mu) = xy + z + \lambda (x^2 + y^2 + z^2 - R^2) + \mu (x + y - 2)
$$

Here:

- λ corresponds to the sphere constraint.
- μ corresponds to the linear relation.

Step 4: Deriving the Necessary Conditions

Compute the partial derivatives:

$$
\frac{\partial \mathcal{L}}{\partial x} = y + 2 \lambda x + \mu = 0
$$

\[
\frac{\partial \mathcal{L}}{\partial y} = x + 2 \lambda y + \mu = 0
\]
\[
\frac{\partial \mathcal{L}}{\partial z} = 1 + 2 \lambda z = 0
\]
\[
\frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + y^2 + z^2 - R^2 = 0
\]
\[
\frac{\partial \mathcal{L}}{\partial \mu} = x + y - 2 = 0
\]

Step 5: Solving the System of Equations

From the derivatives with respect to \(x\) and \(y\):

\[
y + 2 \lambda x + \mu = 0 \quad (1)
\]
\[
x + 2 \lambda y + \mu = 0 \quad (2)
\]

Subtract (2) from (1):

\[
(y - x) + 2 \lambda (x - y) = 0
\]

which simplifies to:

\[
(y - x) + 2 \lambda (x - y) = 0
\]
\[
(y - x) - 2 \lambda (y - x) = 0
\]

Factoring:

\[
(y - x)(1 - 2 \lambda) = 0
\]

This yields two cases:

- Case 1: \(y - x = 0 \Rightarrow y = x\)
- Case 2: \(1 - 2 \lambda = 0 \Rightarrow \lambda = \frac{1}{2}\)

Case 1: \(y = x\)

Using the linear constraint \(x + y = 2\):

\[
x + x = 2 \Rightarrow 2x = 2 \Rightarrow x = 1
\]
\[
y = 1
\]

From the derivative with respect to \(z\):

\[
1 + 2 \lambda z = 0 \Rightarrow 2 \lambda z = -1
\]

- If \(\lambda \neq 0\), then:

\[
z = -\frac{1}{2 \lambda}
\]

Now, from the sphere constraint:

\[
x^2 + y^2 + z^2 = R^2
\]
\[
1^2 + 1^2 + z^2 = R^2
\]
\[

$$
2 + z^2 = R^2
$$

$$
z^2 = R^2 - 2
$$

For real z, we need:

$$
R^2 - 2 \geq 0 \Rightarrow R^2 \geq 2
$$

Therefore, the candidate points are:

$$
(x, y, z) = (1, 1, \pm \sqrt{R^2 - 2})
$$

Corresponding function values:

$$
f = xy + z = 1 \times 1 + z = 1 + z
$$

Thus:

$$
f_{\max} = 1 + \sqrt{R^2 - 2}
$$

$$
f_{\min} = 1 - \sqrt{R^2 - 2}
$$

provided $R^2 \geq 2$.

Case 2: $\lambda = \frac{1}{2}$

In this case, from the earlier derivative equations:

$$
y + 2 \cdot \frac{1}{2} x + \mu = 0 \Rightarrow y + x + \mu = 0
$$

\]
\[
x

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