Find The Maximum And The Minimum Values Of F(x,y,z) = 5x - 4y + 8z On The Sphere X + Y2 + 2 = 105 The

Find The Maximum And The Minimum Values Of F(x,y,z) = 5x - 4y + 8z On The Sphere X + Y2 + 2 = 105 The problem is a classic example of optimization in multivariable calculus, where the goal is to find the extremal (maximum and minimum) values of a given function subject to a constraint. Such problems often involve the application of methods like Lagrange multipliers, which provide a systematic way to handle constraints. In this article, we will explore how to approach this problem step-by-step, understand the underlying principles, and carry out the calculations to find the desired extremal values.

Understanding the Problem

Before delving into the solution, it’s essential to clearly understand the components of the problem.

The Function to Optimize

The function is given by:
$$ F(x, y, z) = 5x - 4y + 8z $$
This is a linear function in three variables, representing a plane in three-dimensional space. The goal is to find its maximum and minimum values constrained to a specific surface.

The Constraint: The Sphere

The constraint is:
$$ x + y^2 + 2 = 105 $$
or equivalently,
$$ x + y^2 = 103 $$
This defines a surface in 3D space, which is a paraboloid-like surface (since the constraint involves y^2) rather than a sphere. Notably, the original problem statement mentions “sphere,” but the given constraint resembles a paraboloid. Assuming a typo, let's clarify:

If the problem intended a sphere, the typical form would be:
$$ x^2 + y^2 + z^2 = R^2 $$
and the given constraint looks more like an ellipsoid or paraboloid.

However, based on the provided constraint $x + y^2 = 103$, the surface is a paraboloid opening along the x-axis.

Note: For consistency, we will proceed assuming the intended surface is $x + y^2 = 103$, which is a paraboloid, as per the given equation.

Important: If the actual problem involves a sphere, the constraint should be of the form $x^2 + y^2 + z^2 = R^2$. Please clarify the constraint if necessary. For now, we proceed with the given equation.

Approach to the Problem

To find the maximum and minimum values of F subject to the constraint, the most common method is the Lagrange multipliers technique.

Method Overview

The method involves:
1. Setting up the Lagrangian function:
$$\mathcal{L}(x,y,z,\lambda) = F(x,y,z) - \lambda (g(x,y,z) - c)$$
where $g(x,y,z)$ is the constraint function and c is a constant.

2. Solving the system of equations derived from the partial derivatives:
$$
\frac{\partial \mathcal{L}}{\partial x} = 0, \quad
\frac{\partial \mathcal{L}}{\partial y} = 0, \quad
\frac{\partial \mathcal{L}}{\partial z} = 0, \quad
\frac{\partial \mathcal{L}}{\partial \lambda} = 0
$$

Step-by-Step Solution

Step 1: Define the Constraint Function

Given the constraint:
$$g(x,y,z) = x + y^2 = 103$$

The goal is to find the extremal values of F on the surface $g(x,y,z) = 103$.

Step 2: Set Up the Lagrangian

The Lagrangian function:
$$
\mathcal{L}(x,y,z,\lambda) = 5x - 4y + 8z - \lambda (x + y^2 - 103)
$$

Step 3: Compute Partial Derivatives

Calculate the derivatives with respect to \(x, y, z\), and \(\lambda\):

- \(\frac{\partial \mathcal{L}}{\partial x} = 5 - \lambda = 0 \Rightarrow \lambda = 5\)

- \(\frac{\partial \mathcal{L}}{\partial y} = -4 - 2\lambda y = 0 \Rightarrow -4 - 2(5) y = 0 \Rightarrow -4 - 10 y = 0\)

- \(\frac{\partial \mathcal{L}}{\partial z} = 8 = 0\)

- \(\frac{\partial \mathcal{L}}{\partial \lambda} = -(x + y^2 - 103) = 0 \Rightarrow x + y^2 = 103\)

Note: The derivative with respect to \(z\) yields \(8 = 0\), which indicates that the function \(F\) depends linearly on \(z\) and is unbounded along the \(z\)-direction unless additional constraints are specified.

In typical optimization problems, if the function is linear and unconstrained in some variables, the maximum and minimum occur at boundary points.

Alternatively, if the problem is about the maximum and minimum of \(F\) with respect to the variables constrained on the surface, and \(z\) is unconstrained, then the extremal values may not exist unless bounded.

Clarification:
If the original problem involved the sphere \(x^2 + y^2 + z^2 = R^2 \), the approach would be similar but with a sphere constraint, which is more standard in such optimization problems.

Assuming the problem involves the sphere \(x^2 + y^2 + z^2 = R^2 \), let's proceed with this typical scenario. For demonstration, suppose the sphere is:

\[x^2 + y^2 + z^2 = 105 \]

Revised Problem: Find the maximum and minimum of \(F(x,y,z) = 5x - 4y + 8z\) on the sphere \(x^2 + y^2 + z^2 = 105\)

Step 1: Set Up the Lagrangian

\[
\mathcal{L}(x,y,z,\lambda) = 5x - 4y + 8z - \lambda (x^2 + y^2 + z^2 - 105)
\]

Step 2: Find Partial Derivatives

\[
\frac{\partial \mathcal{L}}{\partial x} = 5 - 2\lambda x = 0 \Rightarrow 2\lambda x = 5 \Rightarrow x = \frac{5}{2\lambda}
\]

\[
\frac{\partial \mathcal{L}}{\partial y} = -4 - 2\lambda y = 0 \Rightarrow 2\lambda y = -4 \Rightarrow y = -\frac{4}{2\lambda} = -\frac{2}{\lambda}
\]

\[
\frac{\partial \mathcal{L}}{\partial z} = 8 - 2\lambda z = 0 \Rightarrow 2\lambda z = 8 \Rightarrow z = \frac{8}{2\lambda} = \frac{4}{\lambda}
\]

Step 3: Use the Constraint to Find \(\lambda\)

Substitute \(x, y, z\) into the sphere equation:
\[
x^2 + y^2 + z^2 = 105
\]

\[
\left(\frac{5}{2\lambda}\right)^2 + \left(-\frac{2}{\lambda}\right)^2 + \left(\frac{4}{\lambda}\right)^2 = 105
\]

Compute each term:
\[
\frac{25}{4\lambda^2} + \frac{4}{\lambda^2} + \frac{16}{\lambda^2} = 105
\]

Combine:
\[
\frac{25}{4\lambda^2} + \frac{4}{\lambda^2} + \frac{16}{\lambda^2} = \frac{25}{4\lambda^2} + \frac{4 + 16}{\lambda^2} = \frac{25}{4\lambda^2} + \frac{20}{\lambda^2}
\]

Express with common denominator:
\[

\frac{25}{4\lambda^2} + \frac{80}{4\lambda^2} = \frac{105}{4\lambda^2}
\]

Set equal to 105:
\[
\frac{105}{4\lambda^2} = 105
\]

Multiply both sides by \(4\lambda^2\):
\[
105 = 105 \times 4 \lambda^2
\]
\[
105 = 420 \lambda^2
\]

Frequently Asked Questions

How do I find the maximum and minimum values of the function F(x,y,z) = 5x - 4y + 8z subject to the sphere constraint x + y^2 + 2 = 105?

To find the extrema, set up a Lagrange multiplier problem with the constraint g(x,y,z) = x + y^2 + 2 - 105 = 0. Form the Lagrangian L = 5x - 4y + 8z + λ(x + y^2 + 2 - 105), then take partial derivatives and solve the system to find critical points, which give the maximum and minimum values.

What are the steps to solve for the maximum and minimum of F(x,y,z) on the given sphere?

First, write the constraint as g(x,y,z) = x + y^2 + 2 - 105 = 0. Next, set up the Lagrangian and find the partial derivatives with respect to x, y, z, and λ. Solve the resulting equations simultaneously to find candidate points, then evaluate F at these points to determine maxima and minima.

Can the method of Lagrange multipliers be used directly on the surface defined by x + y^2 + 2 = 105?

Yes, the method of Lagrange multipliers is suitable for finding extrema of a function constrained to a surface defined by an equation like x + y^2 + 2 = 105. It involves introducing a multiplier λ and solving the system of equations derived from the gradients.

Is the function F(x,y,z) = 5x - 4y + 8z linear, and how does this affect finding its maximum and minimum on the sphere?

Yes, F(x,y,z) is a linear function. When optimizing a linear function over a bounded surface like a sphere, extrema occur at boundary points or critical points found via Lagrange multipliers. The linearity simplifies the derivative calculations but requires solving the constraint equations to find exact values.

What is the significance of the constraint $x + y^2 + 2 = 105$ in the optimization problem?

The constraint defines the surface (a sphere-like surface) on which the function F(x,y,z) is optimized. It limits the domain, ensuring that the maximum and minimum values of F are found only at points lying on this surface, which affects the solution process via the Lagrange multiplier method.

Additional Resources

Finding the maximum and minimum values of a multivariable function constrained to a surface is a fundamental problem in calculus, with applications spanning physics, engineering, economics, and beyond. In this article, we explore the process of determining the extremal values of the function \(F(x,y,z) = 5x - 4y + 8z \) when restricted to the sphere defined by the equation \(x + y^2 + 2 = 105 \). Through a detailed examination of the problem, we will uncover the systematic approach involving Lagrange multipliers, geometric intuition, and algebraic manipulation, providing insights that are both theoretical and practical in nature.

Understanding the Problem: An Overview

Before delving into the solution methodology, it is essential to understand the nature of the problem and the components involved.

The Objective Function

The function to be optimized is:

\[F(x, y, z) = 5x - 4y + 8z \]

This is a linear function in three variables, which geometrically can be viewed as a family of planes in three-dimensional space. The goal is to find the highest and lowest values of \(F \) on a specific surface.

The Constraint Surface

The surface provided is:

\[x + y^2 + 2 = 105 \]

which simplifies to:

$$ x + y^2 = 103 $$

This is a parabolic cylinder extending infinitely along the z-axis, but with a specific relationship between x and y. Since z does not appear in the constraint, the surface is essentially a “cylinder” in the z-direction, with cross-sections in the xy-plane described by $x + y^2 = 103$.

Key Point: The problem reduces to finding the maximum and minimum of a linear function over a surface that extends infinitely in the z-direction. Because z is unconstrained, the extremal values might be unbounded unless the function's structure constrains the solutions.

Mathematical Approach to the Problem

To find the extremal values of F subject to the constraint, the classical method used is the Lagrange multipliers technique, which allows us to incorporate the constraint into the optimization problem.

Why Use Lagrange Multipliers?

This method is particularly effective when optimizing a function under equality constraints. It transforms the constrained problem into an unconstrained one by introducing a new variable (the Lagrange multiplier) and setting up a system of equations.

Formulating the Lagrangian

Define the constraint function as:

$$ G(x, y) = x + y^2 - 103 = 0 $$

Note that z appears in F but not in G, which implies z is free to vary along the surface for fixed x and y. Therefore, extremal values of F for a fixed z are influenced solely by the x and y components.

The Lagrangian function is:

$$ \mathcal{L}(x, y, z, \lambda) = 5x - 4y + 8z + \lambda (x + y^2 - 103) $$

where λ is the Lagrange multiplier.

Solving the System: Step-by-Step Procedure

The process involves taking partial derivatives of $\mathcal{L}$ concerning each variable and setting them equal to zero.

Step 1: Compute Partial Derivatives

$$
\frac{\partial \mathcal{L}}{\partial x} = 5 + \lambda = 0 \quad \Rightarrow \quad \lambda = -5
$$

$$
\frac{\partial \mathcal{L}}{\partial y} = -4 + 2 y \lambda = 0
$$

$$
\frac{\partial \mathcal{L}}{\partial z} = 8 = 0
$$

$$
\frac{\partial \mathcal{L}}{\partial \lambda} = x + y^2 - 103 = 0
$$

Observation: The derivative with respect to z yields:

$$8 = 0$$

which is impossible, indicating a critical point with respect to z does not exist in the traditional sense. Since z appears only linearly in F, and not in the constraint G, the extremal values of F can be unbounded unless constrained in z.

Implication: The problem, as posed, suggests considering the entire set of points across all z satisfying $x + y^2 = 103$. Because z is unrestricted, F can tend to infinity or negative infinity by choosing $z \to \infty$ or $z \to -\infty$. Therefore, the maximum and minimum values of F are unbounded unless the problem specifically constrains z.

Assumption for a Complete Solution: To find finite extrema, we typically consider the case where z is fixed or constrained, or interpret the problem as seeking extremal values with respect to x and y for any z.

Alternative approach: Focus on the (x, y)-plane, considering z as a free parameter, which does not affect the extremal values of the linear part in x and y.

Analyzing the (x, y) Components

Given the previous derivative analysis, the key is to find the maximum and minimum of:

$$F(x, y) = 5x - 4y$$

subject to:

$$x + y^2 = 103$$

This reduces the problem to a constrained optimization in two variables.

Expressing x in terms of y

From the constraint:

$$x = 103 - y^2$$

Substitute into F:

$$F(y) = 5(103 - y^2) - 4 y = 515 - 5 y^2 - 4 y$$

This is a quadratic function in y:

$$F(y) = -5 y^2 - 4 y + 515$$

Finding the Extremal Values of $F(y)$

Since $F(y)$ is a quadratic with a negative leading coefficient, it opens downward, indicating its maximum is at the vertex.

Vertex of the quadratic:

$$y_{vertex} = - \frac{b}{2a} = - \frac{-4}{2 \times -5} = - \frac{-4}{-10} = - \frac{4}{10} = -0.4$$

Calculate $F(y)$ at $y = -0.4$:

$$F(-0.4) = -5 (-0.4)^2 - 4 (-0.4) + 515$$

$$
= -5 (0.16) + 1.6 + 515 = -0.8 + 1.6 + 515 = 0.8 + 515 = 515.8
$$

This is the maximum value of F over the surface.

Finding the minimum value

Because the quadratic opens downward, the minimum occurs at the ends of the feasible y domain. Since the domain is unbounded in y (no restrictions), theoretically, $F(y) \to -\infty$ as $y \to \pm \infty$.

Thus:

- Maximum of F: approximately 515.8 at $y = -0.4$, with the corresponding x:

$$
x = 103 - y^2 = 103 - (-0.4)^2 = 103 - 0.16 = 102.84
$$

- Corresponding point:

$$
(x, y) = (102.84, -0.4)
$$

- The maximum F value is:

$$
F_{max} = 5 \times 102.84 - 4 \times (-0.4) + 8z
$$

which simplifies to:

$$
F_{max} = 514.2 + 1.6 + 8z = 515.8 + 8z
$$

Similarly, the minimum value tends to negative infinity as $y \to \pm \infty$.

Incorporating the z-Variable

Since the original function is:

$$
F(x, y, z) = 5x - 4y + 8z
$$

and the constraint does not restrict z, the maximum and minimum values of F are unbounded unless additional bounds are specified for z.

Implications:

- For any fixed z, the extremal values with respect to \(x

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