Find The Point On The Graph Of The Given Function At Which The Slope Of The Tangent Line Is The Given

Find The Point On The Graph Of The Given Function At Which The Slope Of The Tangent Line Is The Given is a fundamental problem in calculus that involves understanding the relationship between a function's derivative and the slope of its tangent line at a specific point. This type of problem is common in calculus courses and is crucial for analyzing the behavior of functions, especially in optimization, curve sketching, and real-world applications such as physics and engineering. The goal is to determine the exact point(s) on a function where the tangent line has a specified slope, which requires a combination of differentiation techniques and algebraic solving.

In this comprehensive guide, we will explore the step-by-step process to find the point(s) on a function where the tangent line's slope matches a given value. We will cover the fundamental concepts of derivatives, how to interpret the slope of a tangent line, methods for solving such problems, and provide practical examples to solidify understanding. Whether you're a student preparing for exams or a professional applying calculus concepts, this article will serve as a detailed resource.

Understanding the Concept of Tangent Line and Slope

What Is a Tangent Line?

A tangent line to a curve at a specific point is a straight line that touches the curve at that point and has the same instantaneous direction as the curve. In simpler terms, the tangent line "just touches" the curve without crossing it at that point, representing the slope or rate of change of the function at that point.

What Is the Slope of a Tangent Line?

The slope of the tangent line at a point on a function is given by the derivative of the function evaluated at that point. If $y = f(x)$, then the slope of the tangent line at $x = a$ is $f'(a)$. This slope indicates how rapidly the function is increasing or decreasing at that specific point:

- If $f'(a) > 0$, the function is increasing at $x = a$.
- If $f'(a) < 0$, the function is decreasing at $x = a$.
- If $f'(a) = 0$, the point may be a local maximum, minimum, or an inflection point.

Formulating the Problem: Find the Point Where the Slope Equals a Given Value

Suppose you are given:

- A function $f(x)$.
- A specific slope value m.

Your task is to find the point(s) $x = a$ on the graph where the tangent line has a slope of m. This involves:

1. Differentiating the function to find $f'(x)$.
2. Setting $f'(x) = m$.
3. Solving for x to find the point(s) where the slope condition is satisfied.
4. Computing the corresponding y-coordinate by evaluating $f(x)$ at those points.

Step-by-Step Process to Find the Point(s) with a Given Slope

Step 1: Find the Derivative of the Function

The derivative $f'(x)$ provides the slope of the tangent line at any point x. Use differentiation rules (power rule, product rule, quotient rule, chain rule) to compute $f'(x)$.

Step 2: Set the Derivative Equal to the Given Slope

Solve the equation:
$$f'(x) = m$$
for x. This step involves algebraic manipulation and sometimes solving quadratic or higher-degree equations.

Step 3: Solve for the x-Values

Find all real solutions to the equation $f'(x) = m$. These solutions correspond to the points where the tangent line has the desired slope.

Step 4: Find Corresponding y-Coordinates

Substitute each x value back into the original function:
$$y = f(x)$$
to find the point(s) (x, y).

Step 5: Verify the Solutions

Check the solutions to ensure they are valid within the context of the problem, especially if the function has restrictions or undefined points.

Practical Examples

Example 1: Polynomial Function

Suppose $f(x) = x^3 - 3x + 2$, and you need to find the point where the tangent line has a slope of 3.

Solution:

1. Differentiate:
$$f'(x) = 3x^2 - 3$$

2. Set equal to 3:
$$3x^2 - 3 = 3$$
$$3x^2 = 6$$
$$x^2 = 2$$
$$x = \pm \sqrt{2}$$

3. Find y values:
- For $x = \sqrt{2}$:
$$y = (\sqrt{2})^3 - 3(\sqrt{2}) + 2 = 2\sqrt{2} - 3\sqrt{2} + 2 = -\sqrt{2} + 2$$
- For $x = -\sqrt{2}$:
$$y = (-\sqrt{2})^3 - 3(-\sqrt{2}) + 2 = -2\sqrt{2} + 3\sqrt{2} + 2 = \sqrt{2} + 2$$

Points:

$$ (\sqrt{2}, -\sqrt{2} + 2) $$

$$ (-\sqrt{2}, \sqrt{2} + 2) $$

Example 2: Trigonometric Function

Suppose $f(x) = \sin x$, and the goal is to find where the tangent slope is 0 (horizontal tangent).

Solution:

1. Derivative:

$$ f'(x) = \cos x $$

2. Set equal to 0:

$$ \cos x = 0 $$

$$ x = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z} $$

3. Find y:

$$ y = \sin x $$

- For $x = \frac{\pi}{2}$:

$$ y = 1 $$

- For $x = \frac{3\pi}{2}$:

$$ y = -1 $$

- And so on for other values of n.

Points:

$$ \left(\frac{\pi}{2} + n\pi, \pm 1 \right) $$

Special Cases and Additional Considerations

Multiple Points with the Same Slope

It is common for a function to have multiple points where the tangent line has the same slope. For example, a parabola has two points where the slope matches a given value.

No Solution Cases

Sometimes, the derivative equation may have no real solutions, indicating that the function does not have any points with the specified slope.

Vertical Tangent Lines

If the slope is infinite or undefined, the problem shifts to finding points where the derivative does not exist or tends to infinity, indicating vertical tangent lines.

Implicit Functions and Complex Derivatives

In cases where the function is given implicitly or involves complex derivatives, the approach involves implicit differentiation and more advanced calculus techniques.

Applications of Finding Points with Given Slope

- Curve Sketching: Identifying points of maximum, minimum, or inflection by analyzing where the slope equals zero or another specific value.
- Optimization Problems: Finding points where the rate of change is exactly a certain value to optimize real-world systems.
- Physics: Calculating points where an object's velocity (slope of position vs. time graph) is a specific value.
- Engineering: Designing systems where the rate of change of a parameter must be controlled at certain points.

Summary and Key Takeaways

- To find the point on a graph where the tangent line has a given slope:
1. Differentiate the function.
2. Set the derivative equal to the given slope.
3. Solve for the x-coordinate(s).
4. Find the corresponding y-coordinate(s) by plugging x back into the original function.
- Always verify solutions and consider special cases such as vertical tangents or no solutions.
- Practice with various functions and slopes to strengthen understanding.

Conclusion

Finding the point on a function where the tangent line has a specific slope is a fundamental skill in calculus that combines differentiation with algebraic problem-solving. By understanding the relationship between a function and its derivative, you can efficiently locate these points, providing insights into the function's behavior and enabling applications across science, engineering, and mathematics. With practice, this process becomes an intuitive part of analyzing and interpreting mathematical models, making it an essential tool in the calculus toolkit.

Frequently Asked Questions

How do I find the point on a function where the tangent line has a specified slope?

To find the point, first differentiate the function to get its derivative (slope function). Then, set the derivative equal to the given slope and solve for the x-coordinate. Finally, substitute this x-value into the original function to find the corresponding y-coordinate.

What is the step-by-step process to determine the point at which the tangent line's slope matches a given value?

Step 1: Find the derivative of the function. Step 2: Set the derivative equal to the given slope and solve for x. Step 3: Plug the x-value back into the original function to find y. The point (x, y) is where the tangent has the specified slope.

Can you give an example of finding a point where the tangent slope equals 3 for the function y = x^2?

Yes. First, find the derivative: y' = 2x. Set 2x = 3, so x = 3/2. Then, find y at x = 3/2: y = (3/2)^2 = 9/4. So, the point is (1.5, 2.25).

Why is it important to differentiate the function when finding the point with a specific tangent slope?

Differentiation provides the slope function (derivative), which tells you the slope of the tangent line at any point. Setting this equal to the desired slope allows you to find the corresponding point(s) on the graph.

What if the derivative equation yields multiple solutions when finding the point for a given tangent slope?

Multiple solutions indicate that there are multiple points where the tangent line has that slope. You should evaluate each solution to determine the corresponding points on the graph.

Are there any special cases or functions where finding this point is more complex?

Yes. Functions with implicit forms, piecewise definitions, or involving absolute values can complicate differentiation and solving for the point. In such cases, additional algebraic or calculus techniques may be required.

How does understanding the relationship between derivatives and tangent slopes help in graph analysis?

It helps identify where the graph has specific slopes, such as maxima, minima, or inflection points, and is essential in sketching the behavior and shape of functions accurately.

Additional Resources

Find The Point On The Graph Of The Given Function At Which The Slope Of The Tangent Line Is The Given is a common problem in calculus that combines understanding of derivatives, tangent lines, and algebraic manipulation. Whether you're a student preparing for exams or a professional brushing up on core concepts, mastering this skill enables you to analyze and interpret the behavior of functions more effectively. This guide aims to walk you through the process step-by-step, providing clarity and insight into how to approach such problems with confidence.

Understanding the Concept: The Relationship Between a Function, Its Derivative, and Tangent Lines

Before diving into the problem-solving process, it's essential to grasp the foundational ideas:

- Function $f(x)$: The original curve or graph you're analyzing.
- Tangent Line: A straight line that touches the graph at a single point and has the same slope as the function at that point.
- Derivative $f'(x)$: Represents the slope of the tangent line to the graph of $f(x)$ at any point x.

The problem asks: Given a specific slope m, at what point(s) on the graph of $f(x)$ does the tangent line have that slope?

This essentially involves solving for the x-coordinate(s) where the derivative equals the given slope and then finding the corresponding y-coordinate(s) on the graph.

Step-by-Step Guide to Solving the Problem

1. Identify the Given Function and the Slope

Suppose you're given:

- A function $f(x)$. For example, $f(x) = x^3 - 3x + 2$.
- A specific slope m. For example, $m = 4$.

Your goal is to find the point(s) (x, y) on the graph of $f(x)$ such that the slope of the tangent line at that point is exactly m.

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2. Find the Derivative $f'(x)$

The derivative $f'(x)$ gives the slope of the tangent line at any point x. To compute it:

- Differentiate $f(x)$ using standard rules (power rule, sum rule, etc.).

Example:

For $f(x) = x^3 - 3x + 2$:

$$
f'(x) = 3x^2 - 3
$$

---

3. Set the Derivative Equal to the Given Slope

To find the point(s) where the tangent slope equals m:

$$
f'(x) = m
$$

Using the example:

$$
3x^2 - 3 = 4
$$

---

4. Solve for x

Rearrange the equation and solve for x:

$$
3x^2 = 4 + 3
$$
$$
3x^2 = 7
$$
$$
x^2 = \frac{7}{3}
$$
$$
x = \pm \sqrt{\frac{7}{3}}
$$

This yields two x-values, corresponding to two points on the graph where the tangent line's slope is 4.

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5. Find the Corresponding y-Coordinates

Plug the x-values back into the original function:

$$
y = f(x)
$$

Using the example:

For $x = \sqrt{\frac{7}{3}}$,

$$
y = \left(\sqrt{\frac{7}{3}}\right)^3 - 3 \left(\sqrt{\frac{7}{3}}\right) + 2

\]

Calculate step-by-step:

- \(\left(\sqrt{\frac{7}{3}}\right)^3 = \left(\frac{7}{3}\right)^{3/2} = \left(\frac{7}{3}\right)^{1} \times \left(\frac{7}{3}\right)^{1/2} = \frac{7}{3} \times \sqrt{\frac{7}{3}}\)
- \(-3 \times \sqrt{\frac{7}{3}}\)
- Add 2

Similarly, for \(x = -\sqrt{\frac{7}{3}}\), perform the same calculations.

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6. Write the Coordinates of the Points

The points are:

\[
\left(x, f(x) \right)
\]

for each solution \(x\). These are the points on the graph where the tangent line has the specified slope \(m\).

Additional Tips and Considerations

Handling More Complex Functions

When \(f(x)\) is more complex, follow the same steps but be prepared for:

- Factoring polynomial equations
- Using quadratic formulas or other algebraic techniques
- Applying numerical methods if algebraic solutions are complicated or impossible

Verifying the Solutions

Always verify your solutions:

- Plug the \(x\)-values into \(f'(x)\) to confirm the slope matches \(m\).
- Plug the \(x\)-values into \(f(x)\) to find the corresponding \(y\)-coordinates.

Multiple Solutions

It's common to find multiple points where the tangent line has the same slope. Each corresponds to a different point on the graph, often symmetric or distinct in location.

Practical Example: Step-by-Step Application

Let's work through a complete example to reinforce the concepts.

Given:

- $f(x) = 2x^2 + x$
- Find the point(s) where the tangent line has slope $m = 6$.

Solution:

1. Compute $f'(x)$:

$$
f'(x) = 4x + 1
$$

2. Set $f'(x) = 6$:

$$
4x + 1 = 6
$$
$$
4x = 5
$$
$$
x = \frac{5}{4}
$$

3. Find $f\left(\frac{5}{4}\right)$:

$$
f\left(\frac{5}{4}\right) = 2 \left(\frac{5}{4}\right)^2 + \frac{5}{4} = 2 \times \frac{25}{16} + \frac{5}{4} = \frac{50}{16} + \frac{5}{4}
$$

Convert to common denominator:

$$
\frac{50}{16} + \frac{20}{16} = \frac{70}{16} = \frac{35}{8}
$$

4. Point on the graph:

$$
\boxed{\left(\frac{5}{4}, \frac{35}{8} \right)}
$$

The tangent line at this point has a slope of 6.

Understanding the Role of the Tangent Line Equation

Once we've identified the point(s), it’s often useful to write the equation of the tangent line:

$$
y - y_0 = m (x - x_0)
$$

where (x_0, y_0) is the point found, and m is the given slope.

Continuing the previous example:

$$
y - \frac{35}{8} = 6 \left(x - \frac{5}{4} \right)
$$

This line is tangent to the curve at $\left(\frac{5}{4}, \frac{35}{8} \right)$ and has slope 6.

Summary of the Process

To systematically find the point on a function's graph where the tangent line has a given slope:

1. Differentiate to find $f'(x)$.
2. Set $f'(x) = \text{given slope}$ and solve for x.
3. Substitute each x solution into $f(x)$ to find y.

4. Record the points \(\left(x, f(x) \right)\).

Final Thoughts and Tips

- Always verify your solutions by substituting back into \(f'(x)\).
- Be mindful of the possibility of multiple solutions.
- For complex functions, consider numerical methods or graphing tools.
- Remember that the process is essentially solving an equation derived from the derivative, then plugging back into the original function.

Understanding Find The Point On The Graph Of The Given Function At Which The Slope Of The Tangent Line Is The Given problem empowers you with a critical tool in calculus—analyzing the slope behavior of functions and understanding their geometric properties. With practice, you'll find this process becomes intuitive and invaluable in various applications, from physics to engineering and beyond.

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