Find The Power Series Solution To X^2y''-x(1 X)y' Y=O At X=O With Generalformula For The Coefficients

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In the realm of differential equations, finding solutions in the form of power series is a powerful technique, especially when dealing with equations that are difficult to solve with elementary functions. The differential equation $X^2 y'' - x(1 - X) y' + Y = 0$ presents an interesting case that requires a systematic approach to uncover its solutions near a specific point, commonly at $x = 0$. By employing power series methods, we can derive a general formula for the coefficients of the series, enabling us to express the solution explicitly and analyze its behavior extensively.

Understanding the Differential Equation

Before diving into the solution process, it's crucial to understand the structure of the given differential equation:

$$
X^2 y'' - x(1 - X) y' + Y = 0
$$

Here, the notation suggests a second-order differential equation with variable coefficients. The terms include derivatives of y, multiplied by functions of x, indicating that the solution may be expressible as a power series expansion about $x = 0$.

Key observations:

- The coefficient of y'' is X^2, which vanishes at $x = 0$, indicating a regular singular point at $x=0$.
- The presence of y' and y suggests a linear differential equation.

Our goal is to find a power series solution centered at $x = 0$, that is:

$$
y(x) = \sum_{n=0}^{\infty} a_n x^n
$$

where a_n are the coefficients to be determined.

Formulating the Power Series Solution

To employ the power series method, we proceed with the following steps:

Step 1: Assume a Power Series for $y(x)$

Let:

$$
y(x) = \sum_{n=0}^{\infty} a_n x^n
$$

Then, derivatives are:

$$

$$
y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

$$
y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}
$$

Step 2: Substitute into the Differential Equation

Express each term:

- $X^2 y''$:

$$
x^2 y'' = x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n}
$$

- $-x(1 - X) y'$:

$$
-x (1 - x) y' = -x (1 - x) \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

Expand $(1 - x)$:

$$
-x (1 - x) y' = -x \left(1 - x \right) \sum_{n=1}^{\infty} n a_n x^{n-1} = -x \sum_{n=1}^{\infty} n a_n x^{n-1} + x^2 \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

Simplify:

$$
- \sum_{n=1}^{\infty} n a_n x^{n} + \sum_{n=1}^{\infty} n a_n x^{n+1}
$$

- Y is a constant or a known function of x. Assuming Y is a constant or zero, for simplicity, or if it's a known function, it can be expanded as a power series as well.

For this derivation, we will assume $Y = 0$ to focus on the homogeneous case.

Step 3: Assemble the Series and Form the Recurrence Relation

Putting all parts together:

$$
\sum_{n=2}^{\infty} n(n-1) a_n x^{n} - \sum_{n=1}^{\infty} n a_n x^{n} + \sum_{n=1}^{\infty} n a_n x^{n+1} = 0
$$

Reindex the sums to combine terms with the same power of x:

- For the third sum, replace $n \to n-1$:

$$
\sum_{n=2}^{\infty} (n-1) a_{n-1} x^{n}
$$

Now, the entire equation becomes:

$$
\sum_{n=2}^{\infty} n(n-1) a_n x^{n} - \sum_{n=1}^{\infty} n a_n x^{n} + \sum_{n=2}^{\infty} (n-1) a_{n-1} x^{n} = 0
$$

Note that the sum starting at $n=1$ can be shifted to start at $n=2$:

$$
- 1 \cdot a_1 x^1 = - a_1 x
$$

But since the sum is from $n=1$, to combine with the others, we keep it as is.

Expressing all sums from $n=2$ onwards:

$$
\sum_{n=2}^{\infty} \left[n(n-1) a_n - n a_n + (n-1) a_{n-1} \right] x^{n} - a_1 x = 0
$$

Rearranged:

$$
\sum_{n=2}^{\infty} \left[n(n-1) a_n - n a_n + (n-1) a_{n-1} \right] x^{n} = a_1 x
$$

For the coefficients to satisfy this for all x, the coefficients of each power must be zero:

$$
n(n-1) a_n - n a_n + (n-1) a_{n-1} = 0 \quad \text{for } n \geq 2
$$

And the coefficient of x:

$$
- a_1 = 0 \implies a_1 = 0
$$

The recurrence relation:

$$
\left[n(n-1) - n \right] a_n + (n-1) a_{n-1} = 0
$$

Simplify:

$$
[n^2 - n - n] a_n + (n-1) a_{n-1} = 0
$$

$$
(n^2 - 2n) a_n + (n-1) a_{n-1} = 0
$$

Divide through by $n(n-1)$ (for $n \geq 2$):

$$
\frac{n^2 - 2n}{n(n-1)} a_n + a_{n-1} = 0
$$

Simplify numerator:

\[

$$
n(n - 2) / [n(n-1)] a_n + a_{n-1} = 0
$$

$$
\frac{n - 2}{n-1} a_n + a_{n-1} = 0
$$

Therefore:

$$
a_n = - \frac{n - 1}{n - 2} a_{n-1}
$$

This relation allows us to generate coefficients recursively, given initial conditions.

General Formula for the Coefficients

To find a closed-form expression for a_n, analyze the recurrence:

$$
a_n = - \frac{n - 1}{n - 2} a_{n-1}
$$

Starting from $n=2$:

$$
a_2 = - \frac{2 - 1}{2 - 2} a_1
$$

But note that $n=2$ yields denominator zero, indicating the recurrence is undefined at $n=2$.

Hence, special attention is needed for initial coefficients.

Recall earlier that:

$$
a_1 = 0
$$

and, typically, a_0 is arbitrary (say $a_0 = C$).

Given that $a_1 = 0$, then for $n \geq 2$:

$$
a_2 = - \frac{1}{0} a_1 \quad \text{(undefined)}
$$

This suggests that the recurrence is invalid at $n=2$, indicating the need for an initial term or considering the solution structure carefully. Alternatively, the divergence at $n=2$ hints that the power series solution may be of a particular form, possibly involving logarithmic terms or indicating that the solution is a polynomial or has a different structure.

Summary:

- The initial conditions determine the coefficients.
- For the homogeneous case, with $Y=0$, and assuming $a_1=0$

Frequently Asked Questions

What is the differential equation involved in finding the power series solution around x=0?

The differential equation is $x^2 y'' - x(1 - x) y' + y = 0$.

How do we assume the solution form for power series solutions near x = 0?

We assume a solution of the form $y =$ □ $a_n x^n$, where the sum runs from $n=0$ to infinity.

What is the general approach to find the coefficients a_n in power series solutions?

Substitute the series into the differential equation and equate coefficients of like powers of x to derive a recurrence relation for a_n.

How do initial conditions at x=0 influence the power series solution?

Initial conditions determine the values of a_0 and possibly a_1, which serve as starting points for the recurrence relations.

What is the general formula for the coefficients a_n in this specific differential equation?

The coefficients satisfy the recurrence relation derived from substituting the series into the differential equation, typically of the form a_{n+2} = function of a_n and previous coefficients.

Can you provide the explicit recurrence relation for the coefficients in this problem?

Yes, by substituting $y =$ □ $a_n x^n$ into the differential equation, we obtain a relation such as $(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} + a_n = 0$, which can be rearranged to find a_{n+2} in terms of previous coefficients.

What types of solutions (e.g., regular, singular) are expected at $x=0$ for this differential equation?

Since the equation involves singular points at $x=0$, the solutions are generally Frobenius-type solutions, which may include power series with possible logarithmic terms depending on the indicial equation.

How does the presence of the term $x(1 - x) y'$ affect the power series solution process?

It introduces additional terms in the recurrence relation, requiring careful expansion and matching coefficients to derive the general formula for a_n.

Is it possible to express the power series solution in closed form for this differential equation?

In general, such solutions are expressed as infinite series; closed-form solutions are often not available unless the recurrence relation simplifies into known functions, such as hypergeometric functions.

What is the significance of the general formula for the coefficients in understanding the behavior of solutions near $x=0$?

The general coefficient formula helps analyze convergence, identify special solutions, and understand the nature (regular or singular) of the solutions around $x=0$.

Additional Resources

Power Series Solution to the Differential Equation \(x^2 y'' - x(1 - x) y' + Y = 0 \) at \(x=0 \): A Comprehensive Exploration

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Introduction

When tackling differential equations—especially those with variable coefficients—finding solutions can seem like navigating a complex maze. Among the most powerful tools in a mathematician’s arsenal is the power series method, which allows us to express solutions as infinite sums of powers of the independent variable. This approach is particularly effective near points where the equation might be singular or challenging to handle using standard methods.

In this article, we will delve into the process of finding a power series solution to the differential equation:

$$
x^2 y'' - x(1 - x) y' + Y = 0,
$$

specifically at $x=0$. We will explore how to derive the general formula for the coefficients of the series, interpret the structure of the solution, and understand the implications of the method in solving differential equations with variable coefficients.

Understanding the Differential Equation

Before embarking on the solution, it’s crucial to analyze the given differential equation:

$$
x^2 y'' - x(1 - x) y' + Y = 0.
$$

Key observations:

- The equation involves derivatives of y, with coefficients that are functions of x.
- The presence of x^2 and $x(1 - x)$ suggests potential singularity at $x=0$.
- The term Y appears, but it may be a misprint; in differential equations, the term is usually a function or zero. For the sake of this derivation, assuming the term is zero or a known function, we proceed as if the right side is zero.

Assumption: The standard form of the differential equation is:

$$
x^2 y'' - x(1 - x) y' = 0,
$$

or equivalently, the nonhomogeneous term Y is zero. If Y is a known function, the approach adapts accordingly.

Step 1: Express the Solution as a Power Series

The core idea of the power series method is to assume:

$$
y(x) = \sum_{n=0}^\infty a_n x^n,
$$

where a_n are coefficients to be determined.

Our goal is to substitute this series into the differential equation and find a recurrence relation for a_n.

Step 2: Compute Derivatives

Given:

$$
y(x) = \sum_{n=0}^\infty a_n x^n,
$$

we compute:

$$
y'(x) = \sum_{n=1}^\infty n a_n x^{n-1},
$$

$$
y''(x) = \sum_{n=2}^\infty n(n - 1) a_n x^{n-2}.
$$

Step 3: Substitute into the Differential Equation

Substituting into the original equation:

$$
x^2 y'' - x(1 - x) y' + Y = 0,
$$

we get:

$$
x^2 \left(\sum_{n=2}^\infty n(n-1) a_n x^{n-2} \right) - x(1 - x) \left(\sum_{n=1}^\infty n a_n x^{n-1} \right) + Y = 0.
$$

Simplify each term:

1. First term:

$$
x^2 y'' = \sum_{n=2}^\infty n(n-1) a_n x^{n}.
$$

2. Second term:

$$
x(1 - x) y' = x (1 - x) \sum_{n=1}^\infty n a_n x^{n-1} = (1 - x) \sum_{n=1}^\infty n a_n x^{n}.
$$

Expanding:

$$
(1 - x) \sum_{n=1}^\infty n a_n x^{n} = \sum_{n=1}^\infty n a_n x^{n} - \sum_{n=1}^\infty n a_n x^{n+1}.
$$

Step 4: Reindex and Combine Series

Express the entire equation as a power series:

$$
\sum_{n=2}^\infty n(n-1) a_n x^{n} - \left(\sum_{n=1}^\infty n a_n x^{n} - \sum_{n=1}^\infty n a_n x^{n+1} \right) + Y = 0.
$$

Assuming $Y=0$ (homogeneous case), the equation simplifies to:

$$
\sum_{n=2}^\infty n(n-1) a_n x^{n} - \sum_{n=1}^\infty n a_n x^{n} + \sum_{n=1}^\infty n a_n x^{n+1} = 0.
$$

To combine the series efficiently, express all sums with the same powers of x:

- For the third sum, reindex $n \to n-1$:

$$
\sum_{n=1}^\infty n a_n x^{n+1} = \sum_{n=2}^\infty (n-1) a_{n-1} x^{n}.
$$

Now, write the entire expression:

$$
\boxed{
\sum_{n=2}^\infty n(n-1) a_n x^{n} - \sum_{n=1}^\infty n a_n x^{n} + \sum_{n=2}^\infty (n-1) a_{n-1} x^{n} = 0.
}
$$

Note that the second sum involves $n=1$ to infinity, but at $n=1$:

$$
n=1 \Rightarrow 1 a_1 x^{1}.
$$

So, the entire sum can be written as:

$$
\left[n=1 \text{ term}: - a_1 x \right] + \sum_{n=2}^\infty \left[n(n-1) a_n - n a_n + (n-1) a_{n-1} \right] x^{n} = 0.
$$

Step 5: Derive the Recurrence Relation

For the power series to be identically zero, the coefficients of each power of x must vanish:

- Coefficient of x^1:

$$
- a_1 = 0 \Rightarrow a_1 = 0.
$$

- Coefficients of x^n for $n \ge 2$:

$$
n(n-1) a_n - n a_n + (n-1) a_{n-1} = 0.
$$

Simplify:

$$
[n(n-1) - n] a_n + (n-1) a_{n-1} = 0,
$$

$$
[n^2 - n - n] a_n + (n-1) a_{n-1} = 0,
$$

$$
(n^2 - 2n) a_n + (n-1) a_{n-1} = 0,
$$

$$
a_n = \frac{ - (n-1) a_{n-1} }{ n^2 - 2n }.
$$

Factor the denominator:

$$
n^2 - 2n = n(n - 2),
$$

so the recurrence relation is:

$$
\boxed{
a_n = - \frac{n-1}{n(n - 2)} a_{n-1}, \quad n \ge 2,
}
$$

with initial conditions a_0, a_1 determined from boundary conditions or initial values.

Step 6: General Formula for the Coefficients

To find a closed-form expression for \(a_n \), we observe the recurrence:

\[
a_n = - \frac{n-1}{n(n - 2)} a_{n-1}.
\]

This relation involves factorial and polynomial terms; thus, a product expansion is suitable.

Express \(a_n \) in terms of \(a_0 \) or \(a_1 \):

\[
a_n = a_1 \cdot \prod_{k=2}^n \left(- \frac{k-1}{k(k - 2)} \right),
\]

assuming \(a_1 \) as the initial coefficient.

Factor the product:

\[
a_n = a_1 \cdot (-1)^{n-1} \prod_{k=2}^n \frac{k-1}{k(k - 2)}.
\]

Express the denominator:

\[
k(k - 2) = k^2 - 2k,
\]

and the numerator:

\[

k-1.

\]

To simplify the product, split the denominator:

\[

\prod_{k=2}^n \frac{k-1}{k(k -

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