Find The Solution To The Following Differential Equation: Y" + 16y = 0, Y(O) = 2, Y (0) = 8 = A. Y(t)

Find The Solution To The Following Differential Equation: Y" + 16y = 0, Y(O) = 2, Y (0) = 8 = A. Y(t)

Understanding differential equations is fundamental in applied mathematics, physics, engineering, and many other scientific disciplines. They describe how a quantity changes with respect to another, often time. In this article, we delve into solving a second-order linear homogeneous differential equation with constant coefficients:

$$ Y'' + 16Y = 0 $$

with initial conditions:

$$ Y(0) = 2 \quad \text{and} \quad Y'(0) = 8 $$

(Note: Based on the statement, it appears there may be a typo with the initial conditions, as both are given as $Y(0)$. Typically, for second-order equations, initial conditions are specified as $Y(0)$ and $Y'(0)$. We will proceed assuming the initial conditions are:

$$ Y(0) = 2 $$
$$ Y'(0) = 8 $$

and that "Y (0) = 8 = A" was a typographical error. If you intended to specify an initial derivative, we will incorporate that accordingly.)

Understanding the Differential Equation

What Is a Second-Order Linear Homogeneous Differential Equation?

A second-order linear homogeneous differential equation is a differential equation of the form:

$$ aY'' + bY' + cY = 0 $$

where a, b, and c are constants, and Y is the unknown function of t. The equation under consideration:

$$ Y'' + 16Y = 0 $$

is a special case with $a = 1$, $b = 0$, and $c = 16$.

This type of equation often models systems exhibiting oscillatory behavior, such as mechanical vibrations, electrical circuits, and wave phenomena.

Significance of Initial Conditions

Initial conditions specify the state of the system at $t=0$. For second-order equations, they include:

- The initial position: $Y(0)$
- The initial velocity: $Y'(0)$

These conditions allow us to determine the particular solution from the general solution.

Solving the Differential Equation

Step 1: Write the Characteristic Equation

The first step in solving linear differential equations with constant coefficients is to assume solutions of the form:

$$ Y(t) = e^{rt} $$

Substituting into the differential equation gives:

$$ r^2 e^{rt} + 16 e^{rt} = 0 $$

Dividing through by e^{rt} (which is never zero):

$$ r^2 + 16 = 0 $$

This is the characteristic equation:

$$ r^2 + 16 = 0 $$

Step 2: Find the Roots of the Characteristic Equation

Solve for r:

$$ r^2 = -16 $$

$$ r = \pm \sqrt{-16} = \pm 4i $$

The roots are purely imaginary: $r = \pm 4i$.

Step 3: Write the General Solution

Since roots are complex conjugates $r = \alpha \pm \beta i$, with $\alpha=0$ and $\beta=4$, the general solution for the differential equation is:

$$ Y(t) = C_1 \cos(4t) + C_2 \sin(4t) $$

where C_1 and C_2 are constants to be determined using initial conditions.

Applying Initial Conditions

Step 4: Use $Y(0) = 2$ to Find C_1

At $t=0$:

$$ Y(0) = C_1 \cos(0) + C_2 \sin(0) = C_1 \times 1 + C_2 \times 0 = C_1 $$

Given $Y(0) = 2$:

$$ C_1 = 2 $$

Step 5: Find $Y'(t)$ and Use $Y'(0) = 8$ to Find C_2

Differentiate $Y(t)$:

$$ Y'(t) = -4 C_1 \sin(4t) + 4 C_2 \cos(4t) $$

At $t=0$:

$$ Y'(0) = -4 C_1 \times 0 + 4 C_2 \times 1 = 4 C_2 $$

Given $Y'(0) = 8$:

$$ 4 C_2 = 8 \Rightarrow C_2 = 2 $$

Final Solution

Putting it all together, the specific solution to the differential equation with initial conditions is:

$$ \boxed{
Y(t) = 2 \cos(4t) + 2 \sin(4t)
} $$

This solution describes an oscillatory behavior with amplitude determined by the initial conditions, oscillating at an angular frequency of 4 radians per unit time.

Understanding the Behavior of the Solution

Oscillatory Nature

Since the solution involves sine and cosine functions, it oscillates indefinitely. The amplitude of oscillation is:

$$ \sqrt{(C_1)^2 + (C_2)^2} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2 \sqrt{2} $$

The period T of the oscillation is:

$$ T = \frac{2\pi}{\beta} = \frac{2\pi}{4} = \frac{\pi}{2} $$

Thus, the system completes one oscillation every $\frac{\pi}{2}$ units of time.

Physical Interpretation

This solution could model a simple harmonic oscillator like a mass-spring system with no damping, where the restoring force is proportional to displacement, with the proportionality constant linked to the coefficient in the differential equation.

Summary and Key Points

- The characteristic roots are purely imaginary, indicating oscillatory solutions.

- The general solution combines sine and cosine functions with constants determined by initial conditions.

- Initial conditions \(Y(0) = 2\) and \(Y'(0) = 8\) determine the specific solution constants.

- The solution exhibits simple harmonic motion with amplitude \(2\sqrt{2}\) and period \(\pi/2\).

Extensions and Further Study

- Explore the effects of damping by adding a term like \(bY'\) to the differential equation.
- Study non-homogeneous equations where external forces or inputs are present.
- Investigate systems with varying coefficients for more complex behaviors.

Conclusion

The differential equation \(Y'' + 16Y = 0\) with initial conditions \(Y(0) = 2\) and \(Y'(0) = 8\) models simple harmonic motion. The process involves deriving the characteristic equation, solving for roots, and applying initial conditions to determine the particular solution. The resulting function encapsulates oscillatory behavior, fundamental in understanding wave phenomena and oscillations across scientific disciplines. Mastery of these techniques forms the foundation for analyzing more complex dynamical systems.

Frequently Asked Questions

What is the general solution to the differential equation Y'' + 16Y = 0?

The general solution is $Y(t) = C_1 \cos(4t) + C_2 \sin(4t)$, where C_1 and C_2 are constants.

How do the initial conditions Y(0) = 2 and Y'(0) = 8 help determine the specific solution?

They allow us to solve for the constants C_1 and C_2 by substituting $t = 0$ into the general solution and its derivative.

What is the value of Y(0) given the initial condition?

$Y(0) = 2$, which implies that $C_1 = 2$ since $\cos(0) = 1$ and $\sin(0) = 0$.

How do I find Y'(t) for the differential equation solution?

Differentiate $Y(t) = C_1 \cos(4t) + C_2 \sin(4t)$ to get $Y'(t) = -4C_1 \sin(4t) + 4C_2 \cos(4t)$.

Using the initial condition Y'(0) = 8, how do I find C_2?

Substitute $t = 0$ into $Y'(t)$: $Y'(0) = 4C_2 = 8$, so $C_2 = 2$.

What is the specific solution Y(t) to the differential equation with the given initial conditions?

$Y(t) = 2 \cos(4t) + 2 \sin(4t)$.

Is this differential equation homogeneous, and why?

Yes, because it can be written as $Y'' + 16Y = 0$, which is a homogeneous linear differential equation.

How does the solution to this differential equation relate to simple harmonic motion?

The solution involves sine and cosine functions with constant coefficients, characteristic of simple harmonic motion with angular frequency 4 rad/sec.

Additional Resources

Differential Equations: Unlocking the Secrets of Y" + 16Y = 0

Introduction: The Significance of Differential Equations in Mathematical Analysis

Differential equations are fundamental tools in mathematics, physics, engineering, and numerous scientific disciplines. They describe how a quantity changes over time or space and form the backbone of modeling real-world phenomena—from the oscillations of a pendulum to the flow of heat in a material. Among the myriad types of differential equations, linear homogeneous equations with constant coefficients hold a special place due to their relative simplicity and broad applicability.

Today, we delve into a classic example of such an equation:

$$ Y'' + 16Y = 0 $$

accompanied by initial conditions:

$$ Y(0) = 2, \quad Y'(0) = 8 $$

with the goal of determining the explicit form of $Y(t)$. This problem not only exemplifies the solution techniques for second-order linear differential equations but also provides insight into oscillatory systems, as indicated by the presence of constant coefficients.

Understanding the Differential Equation: $Y'' + 16Y = 0$

What does this equation represent?

This differential equation suggests a system where the second derivative of $Y(t)$—which can be thought of as acceleration—is proportional to the negative of the function itself. Physically, such equations commonly model simple harmonic motion, like a mass-spring system or an LC circuit, where the restoring force is proportional to displacement and acts in the opposite direction.

Key characteristics:

- Type: Homogeneous, linear, second-order differential equation
- Coefficients: Constant
- Nature of solutions: Oscillatory, due to the positive coefficient of Y with a negative sign

---

The General Approach to Solving the Equation

The standard method involves:

1. Formulating the characteristic equation
2. Finding the roots
3. Constructing the general solution
4. Applying initial conditions to find specific constants

Let's explore each step meticulously.

Step 1: Formulate the Characteristic Equation

Replace Y'' with r^2 and Y with the exponential trial solution e^{rt}:

$$
r^2 + 16 = 0
$$

This quadratic equation is the characteristic equation, which determines the form of the solution.

Step 2: Find the Roots of the Characteristic Equation

Solve for r:

$$
r^2 = -16
$$

$$
r = \pm \sqrt{-16} = \pm 4i
$$

These roots are purely imaginary, indicating oscillatory solutions.

Step 3: Construct the General Solution

For roots $r = \pm 4i$, the general solution to the differential equation is:

$$
Y(t) = C_1 \cos(4t) + C_2 \sin(4t)
$$

where C_1 and C_2 are constants determined by initial conditions.

Step 4: Apply Initial Conditions to Determine Constants

Given:

$$
Y(0) = 2
$$
$$
Y'(0) = 8
$$

Calculate $Y(0)$:

$$
Y(0) = C_1 \cos(0) + C_2 \sin(0) = C_1 \times 1 + C_2 \times 0 = C_1
$$

Thus:

$$
C_1 = 2
$$

Next, compute $Y'(t)$:

$$
Y'(t) = -4 C_1 \sin(4t) + 4 C_2 \cos(4t)
$$

Evaluate at $t = 0$:

$$
Y'(0) = -4 C_1 \times 0 + 4 C_2 \times 1 = 4 C_2
$$

Given $Y'(0) = 8$:

\[

4 C_2 = 8 \Rightarrow C_2 = 2
\]

Final Solution:

\[
\boxed{
Y(t) = 2 \cos(4t) + 2 \sin(4t)
}
\]

This explicit formula describes the behavior of the system over time, satisfying both the differential equation and the initial conditions.

Beyond the Solution: Interpreting the Result

The solution indicates a simple harmonic oscillation with angular frequency \(\omega = 4 \). The amplitude and phase of oscillation are determined by the initial conditions:

- Initial displacement: \(Y(0) = 2 \)
- Initial velocity: \(Y'(0) = 8 \)

The oscillatory nature is characterized by:

- Amplitude: \(\sqrt{C_1^2 + C_2^2} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2 \sqrt{2} \)
- Frequency: \(\omega = 4 \)

This reveals a system starting at a displacement of 2 units with a velocity of 8 units/sec, oscillating with a frequency of 4 radians/sec.

Additional Insights: Variations and Applications

1. Phase Shift Representation

The solution can be rewritten in the standard form:

\[
Y(t) = R \cos(4t - \phi)
\]

where:

\[

$$
R = 2 \sqrt{2}
$$

$$
\phi = \arctan \left(\frac{C_2}{C_1} \right) = \arctan(1) = \frac{\pi}{4}
$$

This form emphasizes the phase shift and maximum amplitude of oscillation.

2. Physical Analogies

- Mass-Spring System: The solution models a mass attached to a spring oscillating without damping.
- Electrical Circuit: It describes current or voltage oscillations in an LC circuit.
- Mechanical Vibrations: Any system exhibiting simple harmonic motion, such as pendulums for small angles, follows similar equations.

Common Pitfalls and How to Avoid Them

- Misidentifying the Roots: Remember that imaginary roots lead to sinusoidal solutions, not exponential decay or growth.
- Incorrect Initial Condition Application: Carefully evaluate the derivatives at $t=0$ and substitute correctly.
- Neglecting the Complete Solution: Always include both sine and cosine terms unless initial conditions specify otherwise.

Final Thoughts: The Power of Differential Equations in Science

This detailed exploration underscores how a relatively straightforward second-order differential equation encapsulates complex oscillatory behavior. The solution process—finding characteristic roots, constructing the general solution, and applying initial conditions—demonstrates the elegant structure underlying many physical systems.

Whether you're modeling mechanical vibrations, analyzing electrical circuits, or exploring wave phenomena, mastering such differential equations equips you with a potent analytical tool. The specific example of $Y'' + 16Y = 0$ provides a template for approaching more complex equations, emphasizing the importance of understanding roots, solution forms, and physical interpretations.

In conclusion, the explicit solution:

$$
\boxed{
Y(t) = 2 \cos(4t) + 2 \sin(4t)
}
$$

\]

```
not only solves the differential equation but also offers a window into the
harmonic motions pervasive across scientific disciplines, illustrating the
harmonious blend of mathematics and real-world physics.
```

Find The Solution To The Following Differential Equation Y 16y 0 Y O 2 Y 0 8 A Y T

Find The Solution To The Following Differential Equation Y 16y 0 Y O 2 Y 0 8 A Y T

Related Articles

- Determine The Radius And Interval Of Convergence Of The Following Series... SERIES ANSWERS) . (x-1)"
- Boost Bank Is Holding $100 Million In Deposits, With A 10% Reserve Requirement. As A Result, The Bank
- Explain How Lab Technicians Determine The Refractive Index Of Glass Evidence. How Does This Help With

Back to Home