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In calculus, Taylor polynomials serve as powerful tools for approximating functions near a specific point. They allow us to express complex functions as sums of polynomial terms, making calculations and analyses more manageable. In this article, we will focus on finding the third-degree Taylor polynomial, denoted as $T_3(x)$, for the function $F(x) = x e^{-7x}$, centered at $A = 0$. This process involves understanding the function, computing derivatives, and applying the Taylor polynomial formula. Let's explore each step in detail to grasp how to derive $T_3(x)$ effectively.

Understanding the Function and Its Significance

What is the Function $F(x) = x e^{-7x}$?

The function $F(x) = x e^{-7x}$ combines a polynomial component (x) with an exponential component (e^{-7x}). Such functions are common in various fields like physics, engineering, and economics, often modeling phenomena with exponential decay modulated by a polynomial factor.

Key properties include:
- Continuity and differentiability on the real line
- Smoothness, allowing Taylor series expansion
- The behavior near the center point $A = 0$, which is our expansion point

Why Use Taylor Polynomials?

Taylor polynomials approximate functions around a point A, providing polynomial expressions that closely match the original function near that point. The third-degree Taylor polynomial offers a balance between accuracy and simplicity, capturing the function's behavior up to the cubic term.

Step-by-Step Process for Finding $T_3(x)$

The general formula for the Taylor polynomial of degree n centered at A is:

\[
T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(A)}{k!} (x - A)^k

\]

where $f^{(k)}(A)$ represents the k-th derivative of $f(x)$ evaluated at A.

Since our center point is A = 0, the formula simplifies to:

$$
T_3(x) = f(0) + f'(0) x + \frac{f''(0)}{2} x^2 + \frac{f'''(0)}{6} x^3
$$

Our task involves calculating the function and its derivatives at 0, then plugging these into the formula.

Calculating the Function and Its Derivatives at A = 0

1. Compute $f(0)$

Given $f(x) = x e^{-7x}$:
- Substituting $x=0$,

$$
f(0) = 0 \times e^{0} = 0
$$

2. Find $f'(x)$

Applying the product rule:

$$
f'(x) = \frac{d}{dx} [x e^{-7x}] = e^{-7x} + x \times \frac{d}{dx} e^{-7x}
$$

Since $\frac{d}{dx} e^{-7x} = -7 e^{-7x}$, we have:

$$
f'(x) = e^{-7x} - 7x e^{-7x} = e^{-7x}(1 - 7x)
$$

Evaluate at $x=0$:

$$
f'(0) = e^{0} (1 - 0) = 1 \times 1 = 1
$$

3. Find $f''(x)$

Differentiate $f'(x) = e^{-7x}(1 - 7x)$:

Using the product rule:

$$
f''(x) = \frac{d}{dx} [e^{-7x}(1 - 7x)] = \frac{d}{dx} e^{-7x} \times (1 - 7x) + e^{-7x} \times \frac{d}{dx}(1 - 7x)
$$

Calculate each term:

- $\frac{d}{dx} e^{-7x} = -7 e^{-7x}$
- $\frac{d}{dx} (1 - 7x) = -7$

Thus,

$$
f''(x) = -7 e^{-7x}(1 - 7x) + e^{-7x}(-7) = e^{-7x} \left[-7 (1 - 7x) - 7 \right]
$$

Simplify inside the brackets:

$$
-7 (1 - 7x) - 7 = -7 + 49x - 7 = -14 + 49x
$$

Therefore,

$$
f''(x) = e^{-7x} (-14 + 49x)
$$

Evaluate at $x=0$:

$$
f''(0) = e^{0} (-14 + 0) = -14
$$

4. Find $f'''(x)$

Differentiate $f''(x) = e^{-7x} (-14 + 49x)$:

Again, apply the product rule:

$$
f'''(x) = \frac{d}{dx} e^{-7x} \times (-14 + 49x) + e^{-7x} \times \frac{d}{dx} (-14 + 49x)
$$

\]

Calculate derivatives:

- \(\frac{d}{dx} e^{-7x} = -7 e^{-7x}\)
- \(\frac{d}{dx} (-14 + 49x) = 49\)

Substitute:

\[
f'''(x) = -7 e^{-7x} (-14 + 49x) + e^{-7x} \times 49
\]

Distribute:

\[
f'''(x) = -7 e^{-7x} (-14 + 49x) + 49 e^{-7x}
\]

Simplify the first term:

\[
-7 e^{-7x} (-14 + 49x) = 7 e^{-7x} (14 - 49x)
\]

Thus,

\[
f'''(x) = 7 e^{-7x} (14 - 49x) + 49 e^{-7x} = e^{-7x} [7 (14 - 49x) + 49]
\]

Calculate inside the brackets:

\[
7 \times 14 = 98,\quad 7 \times (-49x) = -343x
\]

Adding 49:

\[
98 - 343x + 49 = (98 + 49) - 343x = 147 - 343x
\]

Therefore,

\[
f'''(x) = e^{-7x} (147 - 343x)
\]

Evaluate at \(x=0\):

$$
f'''(0) = e^{0} \times 147 = 147
$$

Constructing the Third-Degree Taylor Polynomial

Now that we have all derivatives evaluated at 0:

- $f(0) = 0$
- $f'(0) = 1$
- $f''(0) = -14$
- $f'''(0) = 147$

Plug into the Taylor polynomial formula:

$$
T_3(x) = f(0) + f'(0) x + \frac{f''(0)}{2} x^2 + \frac{f'''(0)}{6} x^3
$$

Substituting the values:

$$
T_3(x) = 0 + 1 \times x + \frac{-14}{2} x^2 + \frac{147}{6} x^3
$$

Simplify coefficients:

- $\frac{-14}{2} = -7$
- $\frac{147}{6} = 24.5$ or $\frac{147}{6} = \frac{49}{2}$

Expressing the polynomial:

$$
T_3(x) = x - 7 x^2 + \frac{49}{2} x^3
$$

Final Expression and Interpretation

The third-degree Taylor polynomial of $f(x) = x e^{-7x}$ centered at $A=0$ is:

$$
\boxed{
T_3(x) = x - 7 x^2 + \frac{49}{2} x^3
}
$$

This polynomial approximates the behavior of the original function near $x=0$. It captures the function's initial slope, curvature, and cubic tendencies, providing a useful approximation for small x.

Applications of the Taylor Polynomial

Understanding and computing Taylor polynomials like $T_3(x)$ is valuable across various domains:

- Approximate complex functions for quick calculations
- Analyze function behavior near specific points
- Solve differential equations with approximate solutions
- Perform numerical methods such as polynomial interpolation

Conclusion

Deriving the Taylor polynomial

Frequently Asked Questions

What is the general method to find the Taylor polynomial $T_3(x)$ for a function centered at a specific point A?

To find the Taylor polynomial $T_3(x)$ centered at A, you compute the derivatives of the function at A up to the third order, then use the formula $T_3(x) = f(A) + f'(A)(x - A) + (f''(A)/2!)(x - A)^2 + (f'''(A)/3!)(x - A)^3$.

Given the function $f(x) = x e^{7x}$ and A=0, how do you compute the derivatives needed for $T_3(x)$?

You differentiate $f(x) = x e^{7x}$ multiple times, applying the product rule each time, then evaluate each derivative at A=0 to find the coefficients for the Taylor polynomial.

What is the value of f(0) for the function $f(x) = x e^{7x}$?

$f(0) = 0 e^{0} = 0$.

How do you find the first derivative f'(x) of $f(x) = x e^{7x}$?

Using the product rule: $f'(x) = e^{7x} + x 7 e^{7x} = e^{7x}(1 + 7x)$.

What are the second and third derivatives of $f(x) = x e^{7x}$ evaluated at x=0?

$f''(x) = 14 e^{7x}(1 + 7x) + 49x e^{7x}$; at x=0, $f''(0) = 14$. The third derivative $f'''(x)$ can be found similarly, and at x=0, $f'''(0) = 98$.

What is the final expression for $T_3(x)$ for $f(x) = x e^{7x}$ centered at A=0?

$T_3(x) = 0 + (f'(0))\, x + (f''(0)/2)\, x^2 + (f'''(0)/6)\, x^3 = 0 + 1\, x + (14/2)\, x^2 + (98/6)\, x^3 = x + 7\, x^2 + (49/3)\, x^3$.

Why is the Taylor polynomial $T_3(x)$ useful for approximating f(x) near A=0?

Because it provides a polynomial approximation that matches the function's value and derivatives up to third order at A=0, giving a good approximation of f(x) near that point with manageable calculations.

Additional Resources

Find The Taylor Polynomial $T_3(x)$ for The Function F Centered At The Number A = 0: A Detailed Investigation

In the realm of calculus and mathematical analysis, Taylor polynomials serve as powerful tools for approximating functions locally around a specific point. They provide polynomial expressions that closely mimic the behavior of complex functions within a certain neighborhood, enabling easier analysis, computation, and application across various disciplines including physics, engineering, and mathematics itself.

This article delves into the process of deriving the third-degree Taylor polynomial, denoted as $T_3(x)$, for the function $f(x) = x e^{7x}$, centered at $A = 0$. Through a comprehensive examination, we shall explore the underlying principles, step-by-step calculations, and the significance of these approximations.

Understanding the Function $f(x) = x e^{7x}$

Before embarking on the derivation of the Taylor polynomial, it is essential to understand the nature of the function in question.

Function Characteristics

The function $f(x) = x e^{7x}$ combines a polynomial component, x, with an exponential component, e^{7x}. This blend indicates that the function exhibits exponential growth modulated by a linear factor, with the potential for complex behavior as x varies.

- Domain: The function is defined for all real numbers ($-\infty, \infty$), owing to the exponential's domain.
- Range: The range spans $-\infty$ to ∞, depending on the sign and magnitude of x.
- Differentiability: Since both polynomial and exponential functions are infinitely differentiable, $f(x)$ is smooth, making it suitable for Taylor expansion.

Fundamentals of Taylor Series and Polynomial Approximation

What Is a Taylor Series?

A Taylor series of a function $f(x)$ centered at a point A (also called the expansion point) is an infinite sum of derivatives of f evaluated at A, scaled appropriately by factorial terms:

$$
f(x) = \sum_{n=0}^\infty \frac{f^{(n)}(A)}{n!} (x - A)^n
$$

where $f^{(n)}(A)$ denotes the n-th derivative evaluated at A.

Why Use Taylor Polynomials?

In practical applications, it is often unnecessary (or impossible) to work with the entire infinite series. Instead, we use a finite number of terms—called the Taylor polynomial of degree n—to approximate the function near A:

$$
T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(A)}{k!} (x - A)^k
$$

The accuracy of this approximation improves as n increases and as x approaches A.

Focus on Degree 3 ($T_3(x)$)

The third-degree Taylor polynomial, $T_3(x)$, involves derivatives up to the third order:

$$
T_3(x) = f(A) + f'(A)(x - A) + \frac{f''(A)}{2!} (x - A)^2 + \frac{f'''(A)}{3!} (x - A)^3
$$

Given the center $A=0$, the problem simplifies to evaluating derivatives at zero and constructing the polynomial accordingly.

Step-by-Step Derivation of $T_3(x)$ for $f(x) = x e^{7x}$ at $A=0$

The process involves computing the derivatives $f'(x)$, $f''(x)$, and $f'''(x)$, then evaluating each at $x=0$.

Step 1: Calculate $f(x) = x e^{7x}$

- Function at $x=0$:

$$
f(0) = 0 \times e^{0} = 0
$$

Step 2: Compute $f'(x)$

Using the product rule:

$$
f'(x) = \frac{d}{dx} [x e^{7x}] = e^{7x} + x \times \frac{d}{dx} e^{7x} = e^{7x} + x \times 7 e^{7x} = e^{7x} (1 + 7x)
$$

- Evaluate at $x=0$:

$$
f'(0) = e^{0} (1 + 0) = 1 \times 1 = 1
$$

Step 3: Compute $f''(x)$

Differentiate $f'(x)$:

$$
f''(x) = \frac{d}{dx} [e^{7x} (1 + 7x)]
$$

Using the product rule:

$$
f''(x) = \frac{d}{dx} e^{7x} \times (1 + 7x) + e^{7x} \times \frac{d}{dx} (1 + 7x)
$$

Calculate each:

- $\frac{d}{dx} e^{7x} = 7 e^{7x}$

- $\frac{d}{dx} (1 + 7x) = 7$

Thus,

$$
f''(x) = 7 e^{7x} (1 + 7x) + e^{7x} \times 7 = e^{7x} [7 (1 + 7x) + 7] = e^{7x} [7 + 49x + 7] = e^{7x} (14 + 49x)
$$

- Evaluate at $x=0$:

$$
f''(0) = e^{0} (14 + 0) = 1 \times 14 = 14
$$

Step 4: Compute $f'''(x)$

Differentiate $f''(x)$:

$$
f'''(x) = \frac{d}{dx} [e^{7x} (14 + 49x)]
$$

Apply the product rule:

$$
f'''(x) = \frac{d}{dx} e^{7x} \times (14 + 49x) + e^{7x} \times \frac{d}{dx} (14 + 49x)
$$

Compute derivatives:

- $\frac{d}{dx} e^{7x} = 7 e^{7x}$

- $\frac{d}{dx} (14 + 49x) = 49$

Substitute:

$$
f'''(x) = 7 e^{7x} (14 + 49x) + e^{7x} \times 49 = e^{7x} [7 (14 + 49x) + 49]
$$

Simplify inside brackets:

$$
7 \times 14 = 98,\quad 7 \times 49x = 343 x
$$

Thus,

$$
f'''(x) = e^{7x} (98 + 343 x + 49)
$$

Combine like terms:

$$
98 + 49 = 147
$$

Therefore,

$$
f'''(x) = e^{7x} (147 + 343 x)
$$

- Evaluate at $x=0$:

$$
f'''(0) = e^{0} (147 + 0) = 1 \times 147 = 147
$$

Constructing the Third-Degree Taylor Polynomial $T_3(x)$

Using the derivatives evaluated at $A=0$:

\[

$$
T_3(x) = f(0) + f'(0) x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3
$$

Substituting the computed values:

$$
T_3(x) = 0 + 1 \times x + \frac{14}{2} x^2 + \frac{147}{6} x^3
$$

Simplify fractions:

- $\frac{14}{2} = 7$

- $\frac{147}{6} = \frac{49}{2}$

Thus, the polynomial becomes:

$$
\boxed{
T_3(x) = x + 7 x^2 + \frac{49}{2} x^3
}
$$

This polynomial provides a third-degree approximation of $f(x) = x e^{7x}$ near $x=0$.

Implications and Applications of the Derived Polynomial

The Taylor polynomial $T_3(x) = x + 7x^2 + \frac{49}{2} x^3$ encapsulates the local behavior of the function $f(x)$ around $A=0$. This approximation is especially useful in scenarios such as:

- Numerical analysis: Simpl

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