Find The Volume Of The Largest Rectangular Box In The First Octant With Three Faces In The Coordinate

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Understanding how to maximize the volume of a rectangular box constrained within the coordinate axes is a classic problem in calculus and optimization. This problem involves determining the largest possible rectangular box that can be inscribed in the first octant of the 3D coordinate system, with three of its faces lying along the coordinate planes. Such problems are fundamental in mathematical modeling, engineering, and optimization, providing insight into how constraints influence maximum capacity solutions.

In this comprehensive guide, we will explore the step-by-step process to find the volume of the largest rectangular box within the first octant, given specific boundary conditions. We will discuss the geometric setup, mathematical formulation, optimization techniques, and detailed calculations to arrive at the solution.

Understanding the Geometric and Mathematical Setup

Before delving into the solution, it’s vital to visualize and understand the problem's parameters and constraints.

Defining the Problem Space

- The first octant refers to the portion of 3D space where all coordinates are positive: $x > 0,\ y > 0,\ z > 0$.
- The rectangular box is positioned so that one corner is at the origin $(0,0,0)$.
- The three faces of the box are aligned with the coordinate planes: the xy-plane, yz-plane, and xz-plane.
- The box extends into the positive x, y, and z directions to a point (x, y, z).

Constraints for the Box

- The box's edges are along the axes, with opposite corner at (x, y, z).
- The maximum extents in each coordinate direction are limited by predefined boundaries or functions, depending on the problem specifics.
- For simplicity, assume the box is inscribed under a surface $z = f(x, y)$, with the maximum height at (x, y).

Mathematical Formulation of the Problem

The goal is to maximize the volume V of the rectangular box, which is expressed as:

$$
V = x \times y \times z
$$

Given the constraints, z is often limited by a surface $z = f(x, y)$, so:

$$
V(x, y) = x \times y \times f(x, y)
$$

However, for the classic problem (without an additional surface constraint), the maximum volume is often found in the context of a specific boundary, such as a surface $z = c - x - y$, where c is a positive constant, representing a plane cutting through the axes.

Standard Problem: Maximize Box in a Tetrahedral Region

A common version of this problem involves maximizing the volume of a box inscribed within the region bounded by the coordinate planes and the plane:

$$
x + y + z = c
$$

where $c > 0$.

In this scenario:

- The box's corner at (x, y, z) must satisfy:

$$
x \ge 0,\ y \ge 0,\ z \ge 0,\quad \text{and} \quad x + y + z \le c
$$

- The maximal volume occurs when the corner touches the plane:

$$
z = c - x - y
$$

Thus, the volume becomes:

$$
V(x, y) = x y (c - x - y)
$$

Our goal: maximize $V(x, y)$ over the domain where $x, y \ge 0$ and $x + y \le c$.

Optimization Process to Find the Largest Volume

Now, let's proceed step-by-step to find the maximum volume.

Step 1: Express the Volume Function

Given the previous assumptions:

$$
V(x, y) = x y (c - x - y)
$$

which expands to:

$$
V(x, y) = c x y - x^2 y - x y^2
$$

The domain for (x, y) is:

$$
x \ge 0,\ y \ge 0,\ x + y \le c
$$

\]

Step 2: Find Critical Points Using Partial Derivatives

To locate potential maxima, take partial derivatives of V with respect to x and y:

$$
\frac{\partial V}{\partial x} = c y - 2 x y - y^2
$$

$$
\frac{\partial V}{\partial y} = c x - x^2 - 2 x y
$$

Set these derivatives to zero:

$$
c y - 2 x y - y^2 = 0 \quad (1)
$$

$$
c x - x^2 - 2 x y = 0 \quad (2)
$$

Step 3: Solve the System of Equations

From equation (1):

$$
c y = 2 x y + y^2
$$
$$
c y = y (2 x + y)
$$

If $y \neq 0$, divide both sides by y:

$$
c = 2 x + y
$$

Similarly, from equation (2):

$$
c x = x^2 + 2 x y
$$
$$
c x = x (x + 2 y)
$$

If $x \neq 0$, divide both sides by x:

$$
c = x + 2 y
$$

Now, equate the two expressions for c:

$$
2 x + y = x + 2 y
$$
$$
2 x + y = x + 2 y
$$

Simplify:

$$
2 x - x = 2 y - y
$$
$$
x = y
$$

Now, substitute $x = y$ into the expressions for c:

$$
c = 2 x + y = 2 x + x = 3 x
$$
$$
c = x + 2 y = x + 2 x = 3 x
$$

Both yield the same $c = 3 x$, so:

$$
x = y = \frac{c}{3}
$$

Correspondingly, the height coordinate:

$$

$$
z = c - x - y = c - \frac{c}{3} - \frac{c}{3} = c - \frac{2 c}{3} = \frac{c}{3}
$$

Step 4: Calculate the Maximum Volume

The maximum volume:

$$
V_{\text{max}} = x y z = \left(\frac{c}{3} \right) \left(\frac{c}{3} \right) \left(\frac{c}{3} \right) = \frac{c^3}{27}
$$

This occurs at the point:

$$
\boxed{
\left(\frac{c}{3},\ \frac{c}{3},\ \frac{c}{3} \right)
}
$$

which geometrically corresponds to the inscribed box touching all three coordinate planes and the plane $x + y + z = c$.

Generalizing the Result

While the above derivation is specific to the plane $x + y + z = c$, the approach can be adapted to other boundary surfaces or constraints. The key steps involve:

- Expressing the volume as a function of the variables.
- Applying derivative tests to find critical points.
- Verifying the maximum within the feasible domain.

Additional Considerations and Variations

Depending on the problem's context, different constraints or boundary conditions may modify the solution approach.

Case 1: Box in a Different Surface

Suppose the boundary surface is $z = f(x, y)$, where f is a known function, for example, a paraboloid or an exponential decay. The volume function becomes:

$$
V(x, y) = x y f(x, y)
$$

The optimization process involves:

- Computing partial derivatives.
- Using the chain rule if f is complex.
- Ensuring the solutions lie within the domain constraints.

Case 2: Fixed Boundary Plane $z = c$

If the maximum height is constrained by an upper bound $z = c$, then the problem reduces to maximizing $V = x y c$, with $x, y \le$ some bounds, leading to a different set of solutions.

Case 3: Constraints on x, y, z

Additional constraints such as $x \le a$, $y \le b$, or volume limitations may be included, requiring methods such as Lagrange multipliers or linear programming.

Summary of the Key Results

- The largest inscribed rectangular box in the region bounded by the coordinate planes and the plane $x + y + z = c$ has dimensions:

$$
x = y = z = \frac{c}{3}
$$

-

Frequently Asked Questions

What is the problem of finding the volume of the largest rectangular box in the first octant with three faces on the coordinate planes?

The problem involves determining the maximum volume of a rectangular box situated in the first octant (where $x, y, z \geq 0$) such that its faces lie on the coordinate planes (xy-plane, yz-plane, xz-plane) and its opposite corner is somewhere in space, typically constrained by a given surface or boundary.

How do you set up the variables for maximizing the volume of the box in this problem?

You assign variables x, y, and z to the lengths of the edges of the box along the x, y, and z axes respectively. The volume is then expressed as $V = xyz$, and the goal is to maximize V under the given constraints, often involving a boundary surface like $z = f(x,y)$.

What is the typical constraint used in this problem to find the maximum volume?

A common constraint is a surface such as $z = f(x,y)$, for example, $z = 1 - x - y$, which defines the boundary of the region. The box cannot extend beyond this surface, so the maximum volume is found by maximizing $V = xyz$ subject to the condition $z \leq f(x,y)$.

Which mathematical method is most suitable for solving the problem of finding the maximum volume of the rectangular box?

The method of Lagrange multipliers is most suitable, as it helps optimize the volume function $V = xyz$ subject to the boundary constraint $z = f(x,y)$. Alternatively, setting partial derivatives to zero after expressing z in terms of x and y can also be used.

Can you provide a brief step-by-step process to find the largest volume?

Yes. First, express z in terms of x and y using the boundary constraint. Next, write the volume as $V(x,y) = x\ y\ z(x,y)$. Then, take partial derivatives of V with respect to x and y, set them to zero to find critical points, and verify the maximum. Finally, compute the volume at these points to determine the largest one.

What is an example of a boundary surface constraint often used in this problem?

A common example is the plane z = 1 - x - y, which forms a triangle in the xy-plane and slopes downward, creating a feasible region in the first octant for the box's corner.

Why is this problem considered a classic optimization problem in calculus?

Because it involves maximizing a volume function subject to geometric constraints, illustrating the application of multivariable calculus techniques such as partial derivatives, Lagrange multipliers, and optimization within bounded regions, which are fundamental concepts in calculus.

Additional Resources

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Introduction

In the realm of multivariable calculus and optimization problems, geometric figures constrained within coordinate systems often serve as excellent models for understanding spatial relationships, maximums, and minimums. One such classic problem involves finding the maximum volume of a rectangular box situated in the first octant of a three-dimensional coordinate system, with the added condition that three of its faces lie along the coordinate planes. This problem, while seemingly straightforward, offers rich insights into techniques of calculus—particularly partial derivatives and constrained optimization—and demonstrates the interplay between geometry and algebra.

The problem's significance extends beyond its theoretical elegance; it underpins applications in engineering design, packaging, and resource allocation, where maximizing volume under specific constraints is essential. This comprehensive review explores the problem's formulation, mathematical modeling, solution strategies, and broader implications, aiming to deliver a clear, detailed, and insightful understanding.

Understanding the Geometric Setup

The First Octant and Coordinate Planes

The three-dimensional Cartesian coordinate system divides space into eight regions, known as octants. The first octant is characterized by points where:

- $x \geq 0$,
- $y \geq 0$,
- $z \geq 0$.

In this region, all coordinates are non-negative, making it a natural setting for problems involving physical quantities like volume, since negative lengths are non-physical.

The problem specifies that three faces of the rectangular box lie along the coordinate planes:

- The face on the xy-plane coincides with $z=0$,
- The face on the yz-plane coincides with $x=0$,
- The face on the xz-plane coincides with $y=0$.

This setup ensures that the box is anchored at the origin and extends into the first octant, with its edges aligned along the axes.

The Box's Position and Dimensions

The rectangular box can be described by three positive variables:

- x: the length along the x-axis,
- y: the length along the y-axis,
- z: the length along the z-axis.

Given that the faces are on the coordinate planes, the box's vertices are at:

- The origin point: $(0, 0, 0)$,
- The point: $(x, 0, 0)$,
- The point: $(0, y, 0)$,
- The point: $(0, 0, z)$,
- The opposite vertex: (x, y, z).

Mathematical Formulation of the Problem

Objective: Maximize the Volume

The primary goal is to maximize the volume of the rectangular box, which is given by:

$$
V = x \times y \times z
$$

where $x, y, z \geq 0$.

Constraints

In the classical version of this problem, the constraints often involve additional conditions, such as the box fitting under a surface or within a certain region. For the scenario described — "with three faces in the coordinate" — the main constraint is the geometric restriction that the point (x, y, z) must lie within a certain boundary or under a surface.

However, in this specific problem, the problem statement suggests that the box is unbounded, except for the fact that the faces are along the coordinate planes. To introduce a meaningful maximum, a typical assumption is that the box is limited by a surface, such as a plane:

$$
z = f(x, y)
$$

which represents a boundary surface. This is a common extension in such problems to produce a maximum volume within a bounded region.

For illustration, suppose the boundary surface is a plane:

$$
z = k - a x - b y
$$

where:

- $k, a, b > 0$,

and the maximum volume is constrained to the region where the box can fit under this plane, with the conditions:

$$
x \geq 0, \quad y \geq 0, \quad z = k - a x - b y \geq 0
$$

This introduces a feasible region:

$$
a x + b y \leq k
$$

and the volume becomes:

$$
V(x, y) = x y z = x y (k - a x - b y)
$$

The problem reduces to maximizing $V(x, y)$ over the feasible region.

Analytical Approach to the Optimization Problem

Step 1: Express the Volume Function

Given the boundary surface $z = k - a x - b y$, the volume function becomes:

$$
V(x, y) = x y (k - a x - b y)
$$

which simplifies to:

$$
V(x, y) = k x y - a x^2 y - b x y^2
$$

This is a multivariable function with respect to x and y. Our task is to find the critical points—values of x and y that maximize V—and ensure they lie within the feasible region where $z \geq 0$.

Step 2: Find Critical Points via Partial Derivatives

Calculate the partial derivatives:

$$
\frac{\partial V}{\partial x} = k y - 2 a x y - b y^2
$$

$$
\frac{\partial V}{\partial y} = k x - a x^2 - 2 b x y
$$

Set these derivatives to zero to find critical points:

$$
k y - 2 a x y - b y^2 = 0 \quad (1)
$$

$$
k x - a x^2 - 2 b x y = 0 \quad (2)
$$

These equations can be factored:

$$
y (k - 2 a x - b y) = 0
$$

$$
x (k - a x - 2 b y) = 0
$$

Since $x, y \geq 0$, the trivial solutions where $x = 0$ or $y = 0$ correspond to zero volume, which are not maxima. Therefore, the significant solutions satisfy:

$$
k - 2 a x - b y = 0 \quad (3)
$$

$$
k - a x - 2 b y = 0 \quad (4)
$$

Subtracting (4) from (3):

$$
(2 a x + b y) - (a x + 2 b y) = 0
$$

$$
a x - b y = 0
$$

$$
a x = b y
$$

$$
y = \frac{a}{b} x
$$

Substitute into (3):

$$
k - 2 a x - b \left(\frac{a}{b} x \right) = 0
$$

$$
k - 2 a x - a x = 0
$$

$$
k - 3 a x = 0
$$

$$
x = \frac{k}{3 a}
$$

Then,

$$
y = \frac{a}{b} x = \frac{a}{b} \times \frac{k}{3 a} = \frac{k}{3 b}
$$

Finally, the z-coordinate:

$$
z = k - a x - b y = k - a \times \frac{k}{3 a} - b \times \frac{k}{3 b} = k - \frac{k}{3} - \frac{k}{3} = k - \frac{2k}{3} = \frac{k}{3}
$$

This critical point corresponds to a potential maximum volume, assuming the point lies within the feasible region.

Step 3: Calculate the Maximum Volume

Substitute $x = \frac{k}{3 a}$, $y = \frac{k}{3 b}$, and $z = \frac{k}{3}$ into the volume:

$$
V_{\max} = x y z = \left(\frac{k}{3 a} \right) \left(\frac{k}{3 b} \right) \left(\frac{k}{3} \right) = \frac{k^3}{27 a b}
$$

This provides the maximum volume under the boundary surface $z = k - a x - b y$.

Broader Implications and Variations

Generalization to Other Boundary Surfaces

The problem is versatile and can be extended to different boundary surfaces or constraints, such as:

- Spherical surfaces: For instance, fitting the largest box inside a sphere.
- Ellipsoids or paraboloids: To model more complex physical boundaries.
- Multiple constraints: Combining inequalities to simulate realistic scenarios like maximum capacity, material limitations, or safety margins.

Each variation involves formulating a suitable volume function and applying calculus techniques—partial derivatives, Lagrange multipliers, or geometric reasoning—to find maximum points.

Practical Applications

Understanding how to maximize volume within geometric constraints has numerous applications:

- Packaging Design: Creating optimal containers that minimize material use while maximizing internal volume.
- Manufacturing: Designing components or enclosures with maximum capacity under physical or regulatory constraints

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