Find The Volume Of The Solid Bounded By The Cylinder X2 + Y2 = 4 And The Planes Z = 0, Y + Z = 3. = =

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Calculating the volume of a solid bounded by a cylinder and planes involves understanding the geometry of the shapes involved and setting up appropriate integrals. In this comprehensive guide, we will walk through the problem step-by-step, from understanding the given surfaces to computing the volume using calculus techniques. Whether you're a student preparing for exams or a mathematics enthusiast interested in solid geometry, this article provides an in-depth explanation aligned with SEO best practices for clarity and keyword optimization.

Understanding the Geometric Boundaries

Before diving into the calculations, it is essential to analyze the surfaces that define the solid's boundaries.

The Cylinder: $X^2 + Y^2 = 4$

- This is a circular cylinder aligned along the Z-axis.
- The radius of the cylinder is 2, since $x^2 + y^2 = 4$ implies a circle of radius 2.
- The cylinder extends infinitely in both positive and negative Z-directions unless limited by other surfaces.

The Planes: Z = 0 and Y + Z = 3

- The plane $Z = 0$ is the xy-plane, serving as a lower boundary.
- The plane $Y + Z = 3$ can be rewritten as $Z = 3 - Y$.
- This plane intersects with the cylinder, creating an upper boundary for the solid.

Visualizing the Solid

Understanding the shape helps in setting up the integral:

- The solid exists inside the cylinder $x^2 + y^2 \leq 4$.

- The bottom boundary is the xy-plane $Z = 0$.
- The top boundary is the plane $Z = 3 - Y$, which varies with y.
- The intersection of the cylinder and the plane forms a closed 3D region.

Visual tools such as 3D graphs or software like GeoGebra can help visualize the intersection, but for analytical purposes, coordinate setup and integration are key.

Setting Up the Integral for Volume Calculation

The volume V is given by the triple integral over the bounded region:

$$
V = \iiint_{D} dV
$$

where D is the domain defined by the intersection of the cylinder and planes.

To facilitate integration, it's advantageous to choose coordinate systems that align with the symmetry:

- Cylindrical coordinates (r, θ, z):
- $x = r \cos \theta$
- $y = r \sin \theta$
- $z = z$

This choice simplifies the cylinder boundary $r \leq 2$.

Converting Boundaries to Cylindrical Coordinates

- Cylinder: $r \leq 2$
- Bottom plane: $Z = 0$
- Top plane: $Z = 3 - y = 3 - r \sin \theta$

The domain in cylindrical coordinates is:

- $0 \leq r \leq 2$
- $0 \leq \theta \leq 2\pi$
- $0 \leq z \leq 3 - r \sin \theta$

Formulating the Integral for Volume

The volume integral in cylindrical coordinates becomes:

$$
V = \int_{0}^{2\pi} \int_{0}^{2} \int_{0}^{3 - r \sin \theta} r \, dz \, dr \, d\theta
$$

- The r in the integrand accounts for the Jacobian of the transformation from Cartesian to cylindrical coordinates.

Performing the Integration

1. Integrate with respect to z:

$$
\int_{0}^{3 - r \sin \theta} r \, dz = r (3 - r \sin \theta)
$$

2. Set up the remaining double integral:

$$
V = \int_{0}^{2\pi} \int_{0}^{2} r (3 - r \sin \theta) dr \, d\theta
$$

3. Integrate with respect to r:

$$
V = \int_{0}^{2\pi} \left[\int_{0}^{2} (3r - r^2 \sin \theta) dr \right] d\theta
$$

Calculate the inner integral:

$$
\int_{0}^{2} 3r \, dr = \left[\frac{3r^2}{2} \right]_0^{2} = \frac{3 \times 4}{2} = 6
$$

$$
\int_{0}^{2} r^2 \sin \theta \, dr = \sin \theta \left[\frac{r^3}{3} \right]_0^{2} = \sin \theta \times \frac{8}{3}
$$

Thus,

$$
V = \int_{0}^{2\pi} \left(6 - \frac{8}{3} \sin \theta \right) d\theta

\]

Final Integration over \(\theta \)

- Integrate term-by-term:

\[
V = \int_{0}^{2\pi} 6 \, d\theta - \int_{0}^{2\pi} \frac{8}{3} \sin \theta \, d\theta
\]

- The first integral:

\[
6 \times 2\pi = 12\pi
\]

- The second integral:

\[
\frac{8}{3} \times \left[- \cos \theta \right]_0^{2\pi} = \frac{8}{3} \times (- \cos 2\pi + \cos 0) = \frac{8}{3} \times (-1 + 1) = 0
\]

Therefore,

\[
V = 12\pi - 0 = 12\pi
\]

Conclusion: The Volume of the Solid

The volume of the solid bounded by the cylinder \(x^2 + y^2 = 4 \), the plane \(Z=0 \), and the plane \(Y + Z = 3 \) is:

\[
\boxed{
V = 12\pi
}
\]

This elegant result showcases the symmetry of the problem and the power of cylindrical coordinates in simplifying volume calculations.

Additional Insights and Applications

Understanding how to compute the volume of solids bounded by cylinders and planes has numerous applications in engineering, physics, and mathematics:

- Engineering Design: Calculating material volume for manufacturing parts with cylindrical features.
- Physics: Determining mass or charge distribution in cylindrical objects.
- Mathematics Education: Enhancing spatial visualization and integral calculus skills.

Summary of Key Steps

- Visualize the geometric boundaries.
- Convert the problem into cylindrical coordinates for simplicity.
- Set up the triple integral with appropriate limits.
- Perform successive integrations, starting with the innermost.
- Simplify and evaluate the integrals carefully.
- Confirm the symmetry and bounds are correctly applied.

Final Remarks

Calculating the volume of complex solids bounded by cylinders and planes may seem challenging at first, but with a systematic approach—visualizing the problem, choosing suitable coordinate systems, and carefully setting up integrals—it becomes manageable. The result \(V = 12\pi \) exemplifies the harmony between geometry and calculus, providing a valuable technique for solving related problems in advanced mathematics and engineering.

Keywords: volume calculation, cylinder, bounded solid, triple integral, cylindrical coordinates, geometric boundaries, calculus, volume of cylinder-plane intersection

Frequently Asked Questions

How do I set up the integral to find the volume of the solid

bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 3$?

First, recognize the boundaries: the cylinder $x^2 + y^2 = 4$ (a circle of radius 2), the plane $z = 0$ (the xy-plane), and the plane $y + z = 3$ (which can be rewritten as $z = 3 - y$). To set up the integral, express z in terms of y and integrate over the region in the xy-plane where $0 \leq z \leq 3 - y$ and $x^2 + y^2 \leq 4$. Using cylindrical coordinates simplifies the region, with $x = r \cos \theta$, $y = r \sin \theta$, and r from 0 to 2, θ from 0 to 2π, and z from 0 to $3 - r \sin \theta$.

What is the best coordinate system to evaluate the volume in this problem?

Cylindrical coordinates are most suitable because the boundary involves a circular cylinder $x^2 + y^2 = 4$. These coordinates simplify the integration limits and make the volume calculation more straightforward.

How does the plane $y + z = 3$ affect the limits of integration for z?

The plane $y + z = 3$ can be rewritten as $z = 3 - y$. Since $z \geq 0$, the upper limit for z is $3 - y$, and the lower limit is 0. When integrating, for each fixed y, z varies from 0 up to $3 - y$, which constrains the volume bounded above by this plane.

What is the formula for the volume of the solid bounded by the given surfaces?

The volume V can be expressed as a triple integral: $V = \iiint dV$. In cylindrical coordinates, this becomes $V = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=0}^{3 - r \sin \theta} r \, dz \, dr \, d\theta$. Evaluating this integral yields the volume of the solid.

Are there any symmetry considerations that can simplify the volume calculation?

Yes, the symmetry of the circle $x^2 + y^2 = 4$ and the sine function in $r \sin \theta$ allows us to consider symmetry across axes. Since the region is symmetric about certain axes, sometimes the integral can be simplified by analyzing one sector and multiplying accordingly, but in this case, directly integrating over the full range is straightforward.

What are common pitfalls to avoid when computing this volume?

Common mistakes include incorrect limits for z, failing to convert to cylindrical coordinates properly, or neglecting the dependence of the upper z-bound $(3 - y)$ on y (or r and θ). Ensuring correct bounds after coordinate transformation and careful setup of the integral are crucial for an accurate calculation.

Additional Resources

Find The Volume Of The Solid Bounded By The Cylinder $X^2 + Y^2 = 4$ And The Planes $Z = 0$, $Y + Z = 3$ is a classic problem in multivariable calculus that combines geometric intuition with analytic techniques. This problem involves determining the three-dimensional volume of a solid shape confined within a cylindrical surface and bounded above and below by specified planes. Understanding how to approach such problems is fundamental for students and professionals working in fields like engineering, physics, and mathematics, where modeling complex shapes and calculating their properties often depend on volume integration.

In this comprehensive guide, we will systematically analyze the problem, explore the geometric and algebraic tools necessary for its solution, and walk through the step-by-step process to find the volume of this solid. Whether you're preparing for an exam or seeking to deepen your understanding of multivariable calculus, this detailed breakdown aims to clarify each stage of the process.

Understanding the Geometric Configuration

Before diving into calculations, it's crucial to understand the geometric elements involved:

The Cylinder: \(x^2 + y^2 = 4 \)

- This describes a right circular cylinder centered along the z-axis with radius 2.
- Its cross-sections are circles in the xy-plane with radius 2.
- The cylinder extends infinitely along the z-axis unless bounded by other surfaces.

The Planes:

- Plane \(z = 0 \): The xy-plane, serving as the lower boundary of the solid.
- Plane \(y + z = 3 \): An inclined plane that intersects the cylinder, providing the upper boundary.

Visualizing the Boundaries

To visualize the problem:

- Imagine a cylinder of radius 2 centered at the origin, extending vertically.
- The bottom of the solid is the xy-plane.
- The top boundary is the plane \(y + z = 3 \), which slopes upward and intersects the cylinder.

The goal is to find the volume of the region inside the cylinder, between the xy-plane and the inclined plane.

Step 1: Express the Boundaries in a Suitable Coordinate System

Given the symmetry about the z-axis, cylindrical coordinates are most advantageous for this problem:

$$
\begin{cases}
x = r \cos \theta \\
y = r \sin \theta \\
z = z
\end{cases}
$$

where:

- $r \geq 0$
- $0 \leq \theta \leq 2\pi$
- z varies between the lower and upper boundaries.

Rewriting boundaries in cylindrical coordinates:

- The cylinder: $r \leq 2$.
- The plane $z=0$: remains as is.
- The plane $y + z = 3$:

Since $y = r \sin \theta$,

$$
z = 3 - y = 3 - r \sin \theta
$$

This expression provides the upper boundary of the solid for a given r and θ.

Step 2: Determine the Limits of Integration

Radial coordinate r:

- From the cylinder: $0 \leq r \leq 2$.

Angular coordinate θ:

- Complete rotation: $0 \leq \theta \leq 2\pi$.

Vertical coordinate z:

- Lower boundary: $z = 0$.
- Upper boundary: $z = 3 - r \sin \theta$.

Important: The upper boundary must be above the lower boundary, i.e., $3 - r \sin \theta \geq 0$.

Step 3: Find the Region Where the Upper Boundary is Above the Lower Boundary

Since the lower boundary is at $z=0$, and the upper boundary is at $z=3 - r \sin \theta$, the solid exists where:

$$
3 - r \sin \theta \geq 0
$$

This inequality determines the valid r and θ values where the volume exists.

Handling the inequality:

$$
r \sin \theta \leq 3
$$

Given that $r \leq 2$, and $\sin \theta$ varies between -1 and 1, the maximum value of $r \sin \theta$ in magnitude is $2 \times 1 = 2$, which is less than 3.

Therefore, for all $r \in [0, 2]$ and $\theta \in [0, 2\pi]$, the inequality:

$$
r \sin \theta \leq 3
$$

is always satisfied because:

$$
|r \sin \theta| \leq 2 \times 1 = 2 < 3
$$

and thus,

$$
3 - r \sin \theta \geq 1 > 0
$$

for all valid r, θ.

Conclusion: The upper boundary $z=3 - r \sin \theta$ is always above the xy-plane within the cylinder's domain. Therefore, the entire cylinder is bounded above by this plane.

Step 4: Set Up the Integral for Volume

The volume element in cylindrical coordinates is:

$$
dV = r \, dr \, d\theta \, dz
$$

Thus, the volume V is:

$$
V = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=0}^{3 - r \sin \theta} r \, dz \, dr \, d\theta
$$

Step 5: Perform the Integration

Integrate with respect to z:

$$
\int_{z=0}^{3 - r \sin \theta} r \, dz = r \left[z \right]_0^{3 - r \sin \theta} = r (3 - r \sin \theta)
$$

Now, the volume becomes:

$$
V = \int_{0}^{2\pi} \int_{0}^{2} r (3 - r \sin \theta) \, dr \, d\theta
$$

Step 6: Integrate with respect to r:

$$
\int_{0}^{2} r (3 - r \sin \theta) \, dr = \int_{0}^{2} (3r - r^2 \sin \theta) \, dr
$$

Compute each term separately:

$$
\int_{0}^{2} 3r \, dr = 3 \left[\frac{r^2}{2} \right]_0^{2} = 3 \times \frac{4}{2} = 3 \times 2 = 6
$$

$$
\int_{0}^{2} r^2 \sin \theta \, dr = \sin \theta \left[\frac{r^3}{3} \right]_0^{2} = \sin \theta \times \frac{8}{3}
$$

Putting it together:

$$
\int_{0}^{2} r (3 - r \sin \theta) \, dr = 6 - \frac{8}{3} \sin \theta
$$

Step 7: Integrate with respect to θ:

$$
V = \int_{0}^{2\pi} \left(6 - \frac{8}{3} \sin \theta \right) d\theta
$$

Break into two integrals:

$$
V = \int_{0}^{2\pi} 6 \, d\theta - \frac{8}{3} \int_{0}^{2\pi} \sin \theta \, d\theta
$$

Calculate each:

- $\int_{0}^{2\pi} 6 \, d\theta = 6 \times 2\pi = 12 \pi$
- $\int_{0}^{2\pi} \sin \theta \, d\theta = 0$ (since sine is symmetric over a full period)

Thus,

$$
V = 12 \pi - \frac{8}{3} \times 0 = 12 \pi
$$

Final Answer:

The volume of the solid bounded by the cylinder $x^2 + y^2 = 4$, the plane $z=0$, and the plane $y + z = 3$ is:

$$
\boxed{
V = 12 \pi
}
$$

Additional Insights and Considerations

Symmetry and Simplification

- The symmetry of the cylinder about the z-axis allowed the use of cylindrical coordinates, significantly simplifying the integration process.
- The integral involving $\sin \theta$ vanished over the full rotation due to the periodicity and symmetry of the sine function.

Generalization

- This approach can be adapted to more complex shapes or different bounding planes by carefully analyzing the boundaries and choosing the most suitable coordinate system.
- When the upper boundary depends on r and θ, a similar double integration approach applies, with attention to the inequalities that define the domain.

Visualizing the Result

- The volume 12π corresponds to the portion of the cylinder sliced by the

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