Find The Volume Of The Solid Formed When Revolving The Region Bounded By F(x) = Cos X And G(x) = Sinx

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Understanding the process of calculating the volume of a solid of revolution is a fundamental aspect of integral calculus, often encountered in various fields such as engineering, physics, and mathematics. When you revolve a region bounded by certain functions around an axis, the resulting three-dimensional object's volume can be determined using techniques like the disk method or the washer method.

In this comprehensive guide, we will delve into the problem of finding the volume of the solid formed when revolving the region bounded by the functions $f(x) = \cos x$ and $g(x) = \sin x$. This exploration will include a detailed step-by-step methodology, the mathematical reasoning behind each step, and insights into the geometric interpretation of the problem. Whether you're a student preparing for exams or a professional seeking a refresher, this article aims to provide clarity and depth on this interesting application of integral calculus.

Understanding the Bounded Region

Before proceeding to the calculation, it's essential to understand the region defined by the functions $f(x) = \cos x$ and $g(x) = \sin x$. The key aspects include:

Graphical Representation of the Functions

- The function $f(x) = \cos x$ is a standard cosine wave with a period of 2π, oscillating between -1 and 1.
- The function $g(x) = \sin x$ is a sine wave, also with a period of 2π, oscillating between -1 and 1.

Plotting both functions over a typical interval, such as $[0, 2\pi]$, reveals their points of intersection and the region enclosed.

Points of Intersection

To find where the functions intersect, solve:

$$
\cos x = \sin x
$$

Dividing both sides by $\cos x$ (where $\cos x \neq 0$):

$$
1 = \tan x
$$

Hence,

$$
x = \arctan 1 + k\pi = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}
$$

Within the interval $[0, 2\pi]$, the solutions are:

$$
x = \frac{\pi}{4} \quad \text{and} \quad x = \frac{5\pi}{4}
$$

At these points, the functions intersect, and the bounded region is between these points.

Determining Which Function Is On Top

Between the points of intersection, the relative position of the functions varies:

- For $x \in [0, \pi/4]$:

$\cos x > \sin x$

- For $x \in [\pi/4, 5\pi/4]$:

$\sin x > \cos x$

- For $x \in [5\pi/4, 2\pi]$:

$\cos x > \sin x$

Understanding which function is on top is crucial for setting up the integral correctly.

Choosing the Axis of Revolution

The problem specifies revolving the region around a certain axis. Typically, the axes considered are:

- The x-axis
- The y-axis
- Any other horizontal or vertical line

Since the functions are expressed as functions of x, and the region is bounded between $f(x)$ and $g(x)$, the most common and straightforward approach is to revolve around the x-axis.

Assumption: The region bounded by $f(x) = \cos x$ and $g(x) = \sin x$ is revolved around the x-axis.

Setting Up the Integral for the Volume

The main technique for calculating the volume of a solid of revolution is the disk/washer method.

Disk Method Overview

- When revolving a region around the x-axis, the cross-sectional slices perpendicular to the x-axis are disks.
- The volume of each infinitesimal disk is:

$$
dV = \pi [R(x)]^2 dx
$$

where $R(x)$ is the radius of the disk at point x.

Applying the Disk Method to Our Region

Since the region is bounded between two functions, and the functions switch positions as the top function, the integral must be split at the points of intersection.

Key steps:

1. Identify the intervals where each function is on top.
2. Calculate the volume contributions over each interval, considering the outer radius (top function) and inner radius (bottom function).
3. Sum the volumes over all intervals.

Calculating the Volume

Given the above, the volume V is

$$
V = \int_{a}^{b} \pi \left[(\text{outer radius})^2 - (\text{inner radius})^2 \right] dx
$$

In our case:

- When revolving around the x-axis, the "radius" of the disk at a point x is the value of the function itself, since the region is between the functions and the x-axis.

- If the functions are both positive over the interval, the volume between two functions is obtained using the washer method, where the outer radius is the maximum of the two functions, and the inner radius is the minimum.

Partitioning the Integral

Based on the earlier analysis:

- For $x \in [0, \pi/4]$:

$\cos x \geq \sin x$, so the outer radius is $\cos x$, and the inner radius is $\sin x$.

- For $x \in [\pi/4, 5\pi/4]$:

$\sin x \geq \cos x$, so the outer radius is $\sin x$, and the inner radius is $\cos x$.

- For $x \in [5\pi/4, 2\pi]$:

$\cos x \geq \sin x$, so the outer radius is $\cos x$, and the inner radius is $\sin x$.

Assuming the revolution is about the x-axis, the total volume is:

$$
V = \int_{0}^{\pi/4} \pi [(\cos x)^2 - (\sin x)^2] dx + \int_{\pi/4}^{5\pi/4} \pi [(\sin x)^2 - (\cos x)^2] dx + \int_{5\pi/4}^{2\pi} \pi [(\cos x)^2 - (\sin x)^2] dx
$$

Note that the sign of the difference depends on which function is on top.

Calculating Each Integral

Let's focus on simplifying the integrals:

$$
V = \pi \left(\int_{0}^{\pi/4} [\cos^2 x - \sin^2 x] dx + \int_{\pi/4}^{5\pi/4} [\sin^2 x - \cos^2 x] dx + \int_{5\pi/4}^{2\pi} [\cos^2 x - \sin^2 x] dx \right)
$$

Notice that $\cos^2 x - \sin^2 x = \cos 2x$, based on the double-angle identity.

Similarly, $\sin^2 x - \cos^2 x = -\cos 2x$.

Rewriting the integrals:

$$
V = \pi \left(\int_{0}^{\pi/4} \cos 2x dx - \int_{\pi/4}^{5\pi/4} \cos 2x dx + \int_{5\pi/4}^{2\pi} \cos 2x dx \right)
$$

Evaluating the Integrals

The indefinite integral:

$$
\int \cos 2x dx = \frac{1}{2} \sin 2x + C
$$

Now, compute each definite integral:

1. $\int_{0}^{\pi/4} \cos 2x dx = \frac{1}{2} [\sin 2x]_{0}^{\pi/4} = \frac{1}{2} (\sin \frac{\pi}{2} - \sin 0) = \frac{1}{2} (1 - 0) = \frac{1}{2}$

2. $\int_{\pi/4}^{5\pi/4} \cos 2x dx = \frac{1}{2} [\sin 2x]_{\pi/4}^{5\pi/4} = \frac{1}{2} (\sin \frac{5\pi}{2} - \sin \frac{\pi}{2})$

Evaluate:

$$
\sin \frac{5\pi}{2} = \sin (2\pi + \frac{\pi}{2}) = \sin \frac{\pi}{2} = 1
$$

and

$$
\sin \frac
$$

Frequently Asked Questions

How do I set up the integral to find the volume of the solid formed by revolving the region bounded by f(x) = cos x and g(x) = sin x around the x-axis?

You should determine the points of intersection of the two functions, then set up the integral for the volume as $V = \pi \int [(\cos x)^2 - (\sin x)^2] dx$ over the intersection interval, using the washer method with the outer radius as the larger function and the inner radius as the smaller one.

What are the intersection points of f(x) = cos x and g(x) = sin x within the interval [0, π/2]?

The functions intersect where cos x = sin x, which occurs at x = π/4 within [0, π/2].

How do I determine which function is on top between cos x and sin x in the interval [0, π/2]?

In [0, π/2], cos x decreases from 1 to 0, while sin x increases from 0 to 1. They intersect at x = π/4, where both are equal. For x < π/4, cos x > sin x; for x > π/4, sin x > cos x.

Can I use the washer method to find the volume of the solid formed by revolving this region around the x-axis?

Yes, the washer method is appropriate here. You consider the outer radius as the larger function and the inner radius as the smaller one, integrating over the interval between the intersection points.

What is the value of the volume when revolving the region bounded by cos x and sin x from 0 to π/2 around the x-axis?

The volume is approximately 0.448 cubic units, obtained by evaluating the integral $V = \pi \int_0^{\pi/2} [(\cos x)^2 - (\sin x)^2] dx$.

How do I evaluate the integral for the volume in this problem?

Express $(\cos x)^2$ and $(\sin x)^2$ using the power-reduction formulas: $(\cos x)^2 = (1 + \cos 2x)/2$ and $(\sin x)^2 = (1 - \cos 2x)/2$. Then, set up the integral and integrate term-by-term over [0, π/4] and [π/4, π/2].

What is the significance of the intersection point at x = π/4 in calculating the volume?

The intersection point x = π/4 marks the boundary where the roles of the functions switch in the region; dividing the integral at this point simplifies the calculation by considering which function is on top in each sub-interval.

Is it necessary to consider the symmetry of the functions when calculating the volume?

In this case, symmetry can simplify calculations, but since the region is defined between 0 and π/2, direct integration over this interval with appropriate functions is sufficient. Symmetry considerations are more relevant in other intervals or for different boundaries.

Can this method be generalized to find the volume of regions bounded by other pairs of functions?

Yes, the approach of setting up integrals using the washer or disk method applies broadly. The key steps involve identifying intersection points, determining which function is on top, and setting up the integral accordingly.

Additional Resources

Understanding the Volume of Revolution: An In-Depth Exploration of the Region Bounded by $f(x) = \cos x$ and $g(x) = \sin x$

When delving into the fascinating world of calculus, one of the most captivating topics is the calculation of volumes generated by revolving a region around an axis. This process, known as the method of disks or washers, transforms a two-dimensional area into a three-dimensional solid, opening doors to numerous applications in engineering, physics, and even art. Today, we explore a classic problem: finding the volume of the solid formed when the region bounded by $f(x) = \cos x$ and $g(x) = \sin x$ is revolved about a specified axis.

This article aims to provide an exhaustive, step-by-step analysis of this problem, combining theoretical insights with practical calculation techniques. Whether you are a student striving to master calculus concepts or an expert seeking a refresher, this discussion will deepen your understanding of the interplay between functions and the volumes they generate.

Setting the Stage: Understanding the Region and the Axis of Revolution

Before jumping into calculations, it’s vital to understand the geometric region involved and the axis about which it’s revolved.

The Functions and Their Intersection Points

The two functions under consideration are:

- $f(x) = \cos x$,

- $g(x) = \sin x$.

These are fundamental trigonometric functions with well-understood behavior over their domains. Their graphs are periodic, oscillating between -1 and 1, and they intersect at points where $\cos x = \sin x$.

Finding the intersection points:

$$
\cos x = \sin x \implies \tan x = 1 \implies x = \frac{\pi}{4} + n\pi, \quad n \in \mathbb{Z}
$$

Within the principal interval $0 \leq x \leq \pi/2$, the primary intersection occurs at:

$$
x = \frac{\pi}{4}
$$

At this point:

$$
f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}
$$

Visualizing the region:

Between $x=0$ and $x=\pi/2$, the functions intersect at $x=\pi/4$. For $0 < x < \pi/4$:

$$
\cos x > \sin x
$$

and for $\pi/4 < x < \pi/2$:

$$
\sin x > \cos x
$$

This relationship will influence how we set up the integral, particularly when deciding which function is the "outer" or "inner" radius during revolution.

Choosing the Axis of Revolution

The most common axes for revolution are:

- The x-axis,
- The y-axis,
- Vertical or horizontal lines.

In this scenario, let's consider two primary cases:

1. Revolving around the x-axis.
2. Revolving around the y-axis.

Each case involves different integral setups, and the choice depends on the problem’s context. For this article, we’ll focus on revolving the region around the x-axis, which is a classic problem and offers clear insight into the methods involved.

Methodology: Calculating the Volume of Revolution

The core technique employed here is the washer method, which extends the disk method to handle regions where the bounded area is between two curves.

The Washer Method Explained

When a region between two curves is revolved around a horizontal axis (the x-axis in this case), the volume can be visualized as a stack of washers (disks with holes). The volume V is obtained by integrating the cross-sectional areas along the axis of revolution:

$$
V = \int_{a}^{b} \pi \left[R_{\text{outer}}^2 - R_{\text{inner}}^2 \right] dx
$$

where:

- a, b are the bounds of the region,
- R_{outer} is the radius of the outer curve,
- R_{inner} is the radius of the inner curve.

In our case, since the region is bounded by $f(x) = \cos x$ and $g(x) = \sin x$, the radii are determined by the functions' values at each x.

Detailed Step-by-Step Calculation

Let's now execute the calculation for the volume when revolving the region between $f(x)$ and $g(x)$ around the x-axis between $x=0$ and $x=\pi/2$.

Step 1: Defining the Bounds and Region

The intersection point $x=\pi/4$ divides the interval into two subregions:

- Region 1: $0 \leq x \leq \pi/4$, where $\cos x \geq \sin x$,
- Region 2: $\pi/4 \leq x \leq \pi/2$, where $\sin x \geq \cos x$.

We will compute the volume for each region separately and then sum the results.

Step 2: Expressing the Radii for Revolution

Since we're revolving around the x-axis:

- The outer radius at x is the maximum of the two functions,
- The inner radius is the minimum of the two functions.

In Region 1 ($0 \leq x \leq \pi/4$):

- Outer radius: $R_{\text{outer}} = \cos x$,
- Inner radius: $R_{\text{inner}} = \sin x$.

In Region 2 ($\pi/4 \leq x \leq \pi/2$):

- Outer radius: $\sin x$,
- Inner radius: $\cos x$.

Step 3: Setting Up the Integral Expressions

The total volume is:

$$
V = V_1 + V_2
$$

where:

$$
V_1 = \pi \int_{0}^{\pi/4} \left[(\cos x)^2 - (\sin x)^2 \right] dx
$$

$$
V_2 = \pi \int_{\pi/4}^{\pi/2} \left[(\sin x)^2 - (\cos x)^2 \right] dx
$$

Note that the integrand is the difference of squares, which simplifies via trigonometric identities.

Step 4: Simplify the Integrals Using Trigonometric Identities

Recall the identity:

$$
a^2 - b^2 = (a-b)(a+b)
$$

and, more relevantly, the difference of squares of sine and cosine:

$$
\sin^2 x - \cos^2 x = -\cos 2x
$$

This simplifies the integrals significantly.

Therefore:

$$
V_1 = \pi \int_{0}^{\pi/4} -\cos 2x \, dx
$$

$$
V_2 = \pi \int_{\pi/4}^{\pi/2} \cos 2x \, dx
$$

Step 5: Computing the Integrals

Calculating V_1:

$$
V_1 = -\pi \int_{0}^{\pi/4} \cos 2x \, dx
$$

The integral:

$$
\int \cos 2x \, dx = \frac{1}{2} \sin 2x + C
$$

Thus,

$$
V_1 = -\pi \left[\frac{1}{2} \sin 2x \right]_0^{\pi/4} = -\frac{\pi}{2} \left(\sin \frac{\pi}{2} - \sin 0 \right) = -\frac{\pi}{2} (1 - 0) = -\frac{\pi}{2}
$$

Similarly, V_2:

$$
V_2 = \pi \left[\frac{1}{2} \sin 2x \right]_{\pi/4}^{\pi/2} = \frac{\pi}{2} \left(\sin \pi - \sin \frac{\pi}{2} \right) = \frac{\pi}{2} (0 - 1) = -\frac{\pi}{2}
$$

Final Volume Calculation and Interpretation

Adding the two parts:

$$
V = V_1 + V_2 = -\frac{\pi}{2} + (-\frac{\pi}{2}) = -\pi
$$

Since volume cannot be negative, we take the absolute value:

$$
\boxed{
V = \pi
}
$$

This result indicates that the volume of the solid formed when the region bounded by $f(x) = \cos x$ and \(

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