Find The Volume Of The Solid Generated By Revolving The Region Enclosed By The Triangle With Vertices

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Understanding how to determine the volume of a solid generated by revolving a region around an axis is a fundamental concept in calculus. When the region in question is a triangle with specific vertices, this problem becomes an excellent example to illustrate the application of methods such as the disk, washer, or cylindrical shell techniques. This article provides a comprehensive overview of how to find the volume of the solid formed by revolving the region enclosed by a triangle with vertices, complete with step-by-step guidance, formulas, and practical examples.

Understanding the Basic Concept: Volume of Revolution

Before diving into the specifics of triangles, it's essential to grasp the core idea of volume of revolution.

What Is a Solid of Revolution?

A solid of revolution is a three-dimensional object created when a two-dimensional region in the plane is rotated about a line (axis). The shape of the resulting solid depends on the shape of the region and the axis of rotation.

Common Methods to Find Volumes

- Disk Method: Used when the cross-sections perpendicular to the axis are disks.
- Washer Method: Applied when the region is hollowed out, creating washers with inner and outer radii.
- Cylindrical Shell Method: Suitable when the shell method simplifies the integration, especially for revolution around vertical or horizontal axes.

Identifying the Region Enclosed by the Triangle

The first step in solving the problem is clearly defining the triangle's vertices and the region they form.

Vertices of the Triangle

Suppose the triangle has vertices at points:
- $A(x_1, y_1)$
- $B(x_2, y_2)$
- $C(x_3, y_3)$

For example, consider a triangle with vertices:
- $A(0, 0)$
- $B(4, 0)$
- $C(2, 3)$

This triangle is bounded by three lines, which we need to understand to set up the integral correctly.

Equations of the Triangle's Sides

Determine the equations of the lines forming the triangle's sides:

- Line AB: Between $A(0, 0)$ and $B(4, 0)$, which is simply $y=0$.
- Line AC: Between $A(0, 0)$ and $C(2, 3)$. Slope $m_{AC} = (3-0)/(2-0) = 3/2$. Equation: $y = \frac{3}{2}x$.
- Line BC: Between $B(4, 0)$ and $C(2, 3)$. Slope $m_{BC} = (3-0)/(2-4) = 3/(-2) = -\frac{3}{2}$. Equation: Using point-slope form:

$y - 0 = -\frac{3}{2}(x - 4)$

Simplify:

$y = -\frac{3}{2}x + 6$

The region enclosed by these lines forms the triangle.

Choosing the Axis of Revolution

The next critical step is selecting the line about which the region will be revolved.

Common Axes of Revolution

- x-axis (horizontal): Revolution around the x-axis.
- y-axis (vertical): Revolution around the y-axis.

- Other lines: For more complex problems, the axis could be any line, but for clarity, focus on the x- or y-axis.

For simplicity, assume the region is revolved around the x-axis in this example.

Setting Up the Integral: Methods and Formulas

Depending on the axis of revolution and the shape of the region, different methods are appropriate.

Using the Disk/Washer Method

- Suitable when the cross-section perpendicular to the axis is a disk or washer.
- The volume is obtained by integrating the area of these cross-sections along the axis.

Volume formula (for revolution around x-axis):

$$
V = \pi \int_{a}^{b} [R(x)]^2 dx
$$

Where:
- $R(x)$ is the radius of the disk at point x, typically the distance from the axis of revolution to the boundary of the region.

Using the Shell Method

- Preferable when the region is more naturally expressed in terms of x or y, especially for revolution around vertical axes.

Volume formula (for revolution around y-axis):

$$
V = 2\pi \int_{a}^{b} x \cdot h(x) dx
$$

Where:
- x is the radius of the shell,
- $h(x)$ is the height (or length) of the shell at x.

Applying the Disk Method to Our Example

Suppose we revolve the triangular region around the x-axis.

Step 1: Find the bounds of integration
- The region runs from $x=0$ to $x=4$.

Step 2: Express the boundary functions
- The upper boundary is given by line AC, $y = \frac{3}{2}x$.
- The lower boundary is $y=0$.

Step 3: Set up the integral
- The radius of the disk at each x is $\frac{3}{2}x$.

$$
V = \pi \int_{0}^{4} \left(\frac{3}{2}x \right)^2 dx = \pi \int_{0}^{4} \frac{9}{4}x^2 dx
$$

Step 4: Compute the integral

$$
V = \pi \cdot \frac{9}{4} \int_{0}^{4} x^2 dx = \frac{9\pi}{4} \left[\frac{x^3}{3} \right]_0^4 = \frac{9\pi}{4} \cdot \frac{4^3}{3}
$$

Calculate:

$$
4^3 = 64
$$

$$
V = \frac{9\pi}{4} \times \frac{64}{3} = 9\pi \times \frac{64}{12} = 9\pi \times \frac{16}{3} = 9\pi \times \frac{16}{3}
$$

Simplify:

$$
V = 3 \times 16 \pi = 48 \pi
$$

Final Volume:

$$
\boxed{V = 48\pi}
$$

This is the volume of the solid generated when the region enclosed by the triangle with vertices at $(0,0)$, $(4,0)$, and $(2,3)$ is revolved about the x-axis.

General Approach for Arbitrary Triangles

For different triangles and axes, the steps involve:

1. Identify the vertices and equations of the sides.
2. Determine the region boundaries and the limits of integration.
3. Choose the appropriate method (disk, washer, shell).
4. Express the radius or height functions in terms of the variable of integration.
5. Set up and evaluate the integral to find the volume.

Practical Tips

- Always sketch the region and axis of revolution to visualize the problem.
- Find the equations of the lines forming the triangle's sides carefully.
- Determine the intersection points to establish the limits of integration.
- Decide between using the disk/washer or shell method based on ease of integration.
- Simplify integrals using substitution or algebraic manipulation for efficient computation.

Applications and Real-World Examples

Understanding how to find the volume of solids generated by revolving triangular regions has numerous applications:

- Manufacturing: Designing objects with rotational symmetry, such as vases or bottles.
- Engineering: Calculating the volume of components formed by rotational cuts.
- Architecture: Modeling domes and other curved structures.
- Mathematics Education: Developing spatial reasoning and integral calculus skills.

Conclusion

Finding the volume of a solid generated by revolving a region, especially a triangle with vertices, involves a systematic approach grounded in calculus techniques. By accurately defining the region, choosing the appropriate method, and carefully setting up the integral, you can compute the volume with confidence. Whether revolving around the x-axis, y-axis, or any other line, mastering these methods is essential for advanced calculus and various engineering and scientific applications.

Remember: The key steps are to understand the shape, select the proper axis and method, express the boundary functions, and evaluate the resulting integral. With practice, solving these problems becomes more intuitive and straightforward.

Frequently Asked Questions

How do you find the volume of a solid generated by revolving a triangle around an axis?

To find the volume, identify the region enclosed by the triangle, determine the axis of revolution, and then apply methods like the disk or washer method by setting up an integral based on the triangle's boundaries and the axis of rotation.

What is the general formula for the volume of a solid of revolution formed by revolving a region around the x-axis?

The volume is given by V = π ∫ [a to b] (radius)^2 dx, where the radius is the distance from the x-axis to the region's boundary, integrated over the interval from a to b.

When revolving a triangle with vertices (0,0), (a,0), and (0,b) around the x-axis, how do you set up the integral?

You express the line connecting (a,0) and (0,b) as y = -(b/a)x + b, then set up the integral $V = \pi \int_0^a [y(x)]^2$ dx, substituting y(x) into the integral accordingly.

What are common methods used to calculate the volume of solids generated by revolving regions bounded by triangles?

Common methods include the disk method, washer method, and shell method, chosen based on the axis of rotation and the shape of the region.

How does changing the axis of revolution affect the calculation of the volume of the solid?

Changing the axis of revolution alters the radius functions used in the integral, which may require switching between disk, washer, or shell methods to correctly set up the volume integral.

Can the volume of a solid generated by revolving a right triangle be found using symmetry?

Yes, symmetry can simplify calculations, especially when revolving around axes that pass through the triangle's vertices or axes of symmetry, reducing the integral's complexity.

What are the key steps to find the volume of a solid formed by revolving a triangle with vertices (0,0), (a,0), and (0,b) about the y-axis?

Express x as a function of y ($x = (a/b)b$), set up the integral for the shell method $V = 2\pi \int_0^b y\, x(y)\, dy$, and evaluate the integral accordingly.

How do you verify your volume calculation for a solid of revolution generated by a triangle?

You can verify by checking units, using alternative methods (like slicing in different orientations), or comparing results with known formulas or geometric approximations to ensure consistency.

Additional Resources

Find The Volume Of The Solid Generated By Revolving The Region Enclosed By The Triangle With Vertices

Understanding how to calculate the volume of a solid generated by revolving a region around an axis is a fundamental concept in calculus. Particularly, when the region is a triangle with specified vertices, the problem involves geometric insight combined with integration techniques. This article aims to guide you through the process step-by-step, providing a clear, detailed, and approachable explanation for solving such problems.

Introduction to Volume by Revolution

Before diving into the specifics of triangles and their vertices, it's essential to understand the general idea of volume generation through revolution. When a planar region is revolved around an axis—usually the x-axis or y-axis—the resulting three-dimensional shape is called a solid of revolution.

Why Calculate These Volumes?

Calculating the volume of such solids has applications in engineering, physics, and even manufacturing, where understanding the capacity or mass of a revolved object is crucial. The process involves setting up an integral that sums up infinitesimal disks or washers—thin slices of the solid—whose combined volume approximates the entire shape.

Methods for Computing Volumes

Two primary techniques are used:

- Disk/Washer Method: Suitable when the region is revolved around an axis parallel to the coordinate axes, especially when the region is bounded by curves.
- Shell Method: Useful when revolving around the y-axis or x-axis, particularly when the region is better described in terms of the other variable.

In the case of a triangle, both methods can be used, but the disk/washer method is often more straightforward if the region's boundaries are well-defined functions.

Defining the Triangle and Its Vertices

Suppose we are given a triangle with vertices at specific points in the coordinate plane. For example:

- Vertex A at (x_1, y_1)
- Vertex B at (x_2, y_2)
- Vertex C at (x_3, y_3)

Example: A Specific Triangle

Let's consider a concrete example for clarity:

- $A(0, 0)$
- $B(4, 0)$
- $C(2, 3)$

This triangle is bounded by:

- The x-axis between $x=0$ and $x=4$,
- The line connecting A and C,
- The line connecting B and C.

Visualizing the Region

Plotting these points reveals a triangle with a base along the x-axis from 0 to 4 and a vertex at $(2, 3)$. The sides are straight lines, making it straightforward to define equations for the boundaries.

Step 1: Find Equations of the Triangle's Sides

To set up the integral for volume, we need the equations of the lines forming the triangle's sides.

Side 1: From $A(0, 0)$ to $C(2, 3)$

- Slope: $m_{AC} = (3 - 0)/(2 - 0) = 3/2$
- Equation: $y = \frac{3}{2}x$

Side 2: From $B(4, 0)$ to $C(2, 3)$

- Slope: $m_{BC} = (3 - 0)/(2 - 4) = 3/(-2) = -3/2$
- Equation passing through $B(4, 0)$:

$$ y - 0 = -\frac{3}{2}(x - 4) $$
$$ y = -\frac{3}{2}x + 6 $$

Base: Along the x-axis from $x=0$ to $x=4$

- $y=0$

Step 2: Determine the Region of Integration

The enclosed region is bounded:

- Below by the x-axis ($y=0$)
- On the left and right by the lines $y=\frac{3}{2}x$ and $y=-\frac{3}{2}x + 6$

The region between $x=0$ and $x=4$ is where the triangle lies.

Key observation: For each x in $[0, 4]$, the top boundary is the lesser of the two lines:

- For x in $[0, 2]$, the top boundary is $y=\frac{3}{2}x$.
- For x in $[2, 4]$, the top boundary is $y=-\frac{3}{2}x + 6$.

The region is therefore divided into two parts, which suggests that setting up the integral in two segments is appropriate.

Step 3: Choosing the Axis of Revolution

The next step is to decide about the axis around which the region is revolved. Common choices are:

- The x-axis ($y=0$)
- The y-axis ($x=0$)
- Any other vertical or horizontal line

For simplicity, let's assume the region is revolved around the x-axis.

Step 4: Setting Up the Integral Using the Disk Method

Since the region is revolved about the x-axis, each cross-section perpendicular to the x-axis is a disk with radius equal to the y-value at that x.

The general formula:

$$
V = \pi \int_{a}^{b} [\text{radius}(x)]^2 dx
$$

Where:

- $\text{radius}(x)$ is the distance from the x-axis to the boundary of the region at x,
- a and b are the bounds along the x-axis.

Step 5: Computing the Volume

Given the boundary functions, the volume is the sum of the volumes of the disks across the two segments:

$$
V = \pi \left(\int_{0}^{2} [y=\frac{3}{2}x]^2 dx + \int_{2}^{4} [y=-\frac{3}{2}x + 6]^2 dx \right)
$$

Calculating each integral separately:

Part 1: $x \in [0, 2]$

\[

$$
V_1 = \pi \int_{0}^{2} \left(\frac{3}{2}x\right)^2 dx = \pi \int_{0}^{2} \frac{9}{4}x^2 dx
$$

$$
V_1 = \pi \times \frac{9}{4} \int_{0}^{2} x^2 dx = \frac{9\pi}{4} \left[\frac{x^3}{3} \right]_0^{2} = \frac{9\pi}{4} \times \frac{8}{3} = \frac{9\pi}{4} \times \frac{8}{3}
$$

$$
V_1 = \frac{9 \times 8 \pi}{4 \times 3} = \frac{72 \pi}{12} = 6 \pi
$$

Part 2: $x \in [2, 4]$

$$
V_2 = \pi \int_{2}^{4} \left(-\frac{3}{2}x + 6\right)^2 dx
$$

Expand the square:

$$
\left(-\frac{3}{2}x + 6\right)^2 = \left(\frac{3}{2}x - 6\right)^2 = \left(\frac{3}{2}x\right)^2 - 2 \times \frac{3}{2}x \times 6 + 36
$$

$$
= \frac{9}{4}x^2 - 18x + 36
$$

Now, integrate term by term:

$$
V_2 = \pi \int_{2}^{4} \left(\frac{9}{4}x^2 - 18x + 36\right) dx
$$

$$
V_2 = \pi \left[\frac{9}{4} \times \frac{x^3}{3} - 18 \times \frac{x^2}{2} + 36x \right]_2^4
$$

Simplify each term:

- $\frac{9}{4} \times \frac{x^3}{3} = \frac{9}{4} \times \frac{x^3}{3} = \frac{3}{4} x^3$
- $-18 \times \frac{x^2}{2} = -9 x^2$

- $36 x$ remains as is.

Compute at the bounds:

At $x=4$:

$$
\frac{3}{4} \times 64 - 9 \times 16 + 36 \times 4 = \frac{3}{4} \times 64 - 144 + 144
$$

$$
= 48 - 144 + 144 = 48
$$

At $x=2$:

$$
\frac{3}{4} \times 8 - 9 \times 4 + 36 \times 2 = 6 - 36 + 72 = 42
$$

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