Find The Volume Of The Solid Obtained By Rotating About The X = Line The Region Bounded By The X-axis

Find The Volume Of The Solid Obtained By Rotating About The X = Line The Region Bounded By The X-axis

Understanding how to find the volume of a solid obtained by rotating a region around a specific line is a fundamental concept in calculus. In particular, when the region is bounded by the x-axis and a curve, rotating it about a vertical line such as x = c (where c is a constant) creates a three-dimensional object whose volume can be determined using the disk/washer method or the cylindrical shell method. This article explores the process step-by-step, providing detailed explanations, formulas, and examples to help you master this important calculus technique.

Introduction to Volume of Solids of Revolution

Calculus allows us to find the volume of three-dimensional objects created by revolving a two-dimensional region around a line. The key idea is to break down the complex volume into infinitesimally small elements, such as disks, washers, or shells, and then sum their volumes using integration.

Types of methods used:
- Disk/Washer Method
- Cylindrical Shell Method

The choice of method depends on the axis of rotation and the shape of the region.

Understanding the Region Bounded by the X-Axis

Before delving into the calculations, it's essential to visualize the region and understand its boundaries.

Typical scenario:
- The region is bounded below by the x-axis (y=0).
- It is enclosed on the sides by a curve y=f(x), where f(x) ≥ 0 within the interval [a, b].
- The region may extend from x=a to x=b, where the function is positive.

This region, when revolved about a line x=c, forms a solid of revolution, the volume of which can be calculated with the appropriate method.

Choosing the Axis of Rotation: The Line x = c

The line $x=c$ is a vertical line, and the position of c relative to the interval $[a, b]$ significantly influences the method:

- If $c > b$ or $c < a$, the region is entirely on one side of the line.
- If c is between a and b, the region is on both sides of the line.

Note: The calculation varies depending on whether the axis of rotation is to the left, right, or between the bounds.

Methods for Calculating the Volume

There are two primary methods:

1. Disk/Washer Method

This method involves slicing the solid perpendicular to the axis of revolution and summing the volumes of disks or washers.

Applicable when:
- The region is bounded by curves $y=f(x)$ and $y=0$
- The axis of rotation is vertical ($x=c$), and the cross-sections are circular disks or washers.

Steps:

1. Express the radius of the disk or washer in terms of x.
2. Write the volume element $dV = \pi [\text{radius}]^2 dx$.
3. Integrate over the interval $[a, b]$.

Note: If the region is rotated around $x=c$, the radius of each disk is $|x - c|$, and the volume element becomes:

$$ dV = \pi [f(x)]^2 dx $$ (if rotating around the x-axis),
or, for rotation around $x=c$:

$$ dV = \pi [\text{distance from } x \text{ to } c]^2 dx = \pi (x - c)^2 dx $$

when the region is bounded by $y=f(x)$.

2. Cylindrical Shell Method

This method involves slicing the region parallel to the axis of revolution into cylindrical shells.

Applicable when:
- The region is bounded horizontally (by y) and rotated around a vertical line $x=c$.
- The shell method often simplifies calculations when the region is easier to describe as a function of y.

Steps:

1. Identify the height and radius of each shell.
2. Write the volume element as $dV = 2\pi$ (radius) (height) dy.
3. Integrate over the relevant interval.

Note: When rotating around $x=c$, the radius of each shell is $|x - c|$.

Step-by-Step Calculation: Disk/Washer Method

Let's focus on the disk/washer method for the scenario where the region is bounded by $y=f(x) \geq 0$, and rotated about the line $x=c$.

Step 1: Determine the bounds of integration

- The region is between $x=a$ and $x=b$.
- The limits of integration are from $x=a$ to $x=b$.

Step 2: Express the radius

- When rotating around $x=c$, the radius of a disk at point x is $|x - c|$.

Step 3: Write the volume element

- The volume of a thin disk at x is:

$$ dV = \pi [f(x)]^2 dx $$

if rotating around the x-axis.

- When rotating around $x=c$, the cross-sectional area is a washer with inner radius $|x - c|$ and outer radius depending on the shape.

Step 4: Write the integral

- For rotation about $x=c$, the volume is:

$$ V = \int_a^b \pi \left[(x - c)^2 - (\text{inner radius})^2 \right] dx $$

- If the region is between $y=0$ and $y=f(x)$, and the axis is $x=c$, the volume becomes:

$$ V = \int_a^b \pi \left[(x - c)^2 \right] dx $$

assuming the region is fully on one side of the line c.

Step 5: Compute the integral

- Perform the integration analytically, applying the limits.

Example 1: Rotating a Region About $x=2$

Suppose the region bounded by $y=f(x)=x^2$ on $[0, 1]$, rotated about the line $x=2$.

Solution:

- The radius at x is $|x - 2| = 2 - x$ (since $x \leq 1$).
- The volume:

$$V = \int_0^1 \pi (2 - x)^2 dx$$

- Compute:

$$V = \pi \int_0^1 (4 - 4x + x^2) dx$$

$$V = \pi \left[4x - 2x^2 + \frac{x^3}{3} \right]_0^1$$

$$V = \pi \left(4(1) - 2(1)^2 + \frac{1^3}{3} - 0 \right)$$

$$V = \pi \left(4 - 2 + \frac{1}{3} \right) = \pi \left(2 + \frac{1}{3} \right) = \pi \left(\frac{6}{3} + \frac{1}{3} \right) = \frac{7\pi}{3}$$

So, the volume is $\frac{7\pi}{3}$.

Example 2: Rotating About $x=0$

If the same region is rotated about $x=0$:

- Radius at x: $|x - 0| = x$
- Volume:

$$V = \int_0^1 \pi x^2 dx = \pi \frac{x^3}{3} \Big|_0^1 = \frac{\pi}{3}$$

Using the Cylindrical Shell Method for Rotations About x=c

The shell method is particularly advantageous when the region is bounded by $y=f(x)$, and the axis of rotation is vertical ($x=c$). Here's how to proceed:

Step 1: Identify the bounds of x: $[a, b]$.

Step 2: For each x in $[a, b]$, the height of the shell is $f(x)$.

Step 3: The radius of each shell is the horizontal distance from x to $x=c$: $|x - c|$.

Step 4: Write the volume element:

$$ dV = 2\pi \times |x - c| \times f(x) dx $$

Step 5: Set up the integral:

$$ V = \int_a^b 2\pi |x - c| f(x) dx $$

Note: The absolute value can be handled by splitting the integral at $x=c$ if necessary.

Example 3: Shell Method with Rotation Around $x=3$

Suppose $y=\sqrt{x}$ for x in $[0, 4]$, rotated about $x=3$.

Solution:

- The bounds: $x=0$ to $x=4$.
- Shell radius: $|x - 3|$.

- Volume:

$$ V = 2\pi \int_0^4 |x - 3| \times \sqrt{x} dx $$

Split at $x=3$:

$$ V = 2\pi \left(\int_0^3 (3 - x) \sqrt{x} dx + \int_3^4 (x - 3) \sqrt{x} dx \right) $$

Each integral can be evaluated separately using substitution methods.

Special Cases and Tips

- When the region is symmetric about the line $x=c$, calculations may simplify.
- Always verify the limits and the position of the region relative to the axis.
- Be cautious with the absolute value in the shell method; split integrals when

Frequently Asked Questions

What is the general method to find the volume of a solid obtained by rotating a region about the line x = a?

The general method involves using the disk or washer method, where you integrate the cross-sectional areas perpendicular to the axis of rotation, typically setting up the integral with respect to y or x depending on the region.

How do you set up the integral for finding the volume when rotating a region bounded by the x-axis about the line x = a?

You identify the bounds of the region along the x-axis, express the radius of the disks or washers in terms of x, and then integrate π times the square of the radius over the interval to find the volume.

What is the significance of choosing the correct axis of rotation when calculating volume?

Choosing the correct axis of rotation ensures the correct formulation of the radius function and limits, which directly impacts the accuracy of the volume calculation.

Can you explain how the washer method applies when rotating a region about the line x = a?

Yes, the washer method involves subtracting the volume of the inner circle (if any) from the outer circle's volume at each slice, integrating these washers along the interval to find the total volume.

What are common mistakes to avoid when setting up the integral for this type of volume problem?

Common mistakes include incorrect limits of integration, misidentifying the radius function, and forgetting to square the radius or include π in the integral.

How does the position of the line x = a relative to the region affect the volume calculation?

The position determines whether the radius is measured from x = a to the region boundary and influences the limits of integration and the radius function used.

Is it necessary to convert the region into a different coordinate system before computing the volume?

Not necessarily; most problems can be solved in Cartesian coordinates, but sometimes changing to y-coordinates or using symmetry can simplify the integration.

What are some real-world applications of finding volumes of solids obtained by rotation around x = a?

Applications include designing objects like tanks, bottles, and mechanical parts, as well as solving problems in physics and engineering involving rotational bodies.

Additional Resources

Find The Volume Of The Solid Obtained By Rotating About The X = Line The Region Bounded By The X-axis

Understanding the calculation of volume generated by revolving a region around a specific line is fundamental in calculus, especially in the context of volumes of revolution. This problem, which involves rotating a bounded region around the vertical line $x = a$, offers a fascinating exploration of integral calculus, geometric intuition, and the application of various methods such as the disk, washer, and shell methods. It combines analytical rigor with geometric visualization, providing a rich learning experience for students and professionals alike.

In this comprehensive article, we will delve into this problem systematically, exploring the underlying concepts, formulating the mathematical expressions, and demonstrating step-by-step solutions. Whether you are a student seeking to master the techniques or a mathematician interested in the nuances of volume calculations, this review aims to provide clarity, depth, and insightful analysis.

Understanding the Geometric Context

Before diving into calculations, it's essential to grasp the geometric scenario at hand. The problem involves a region in the xy-plane bounded by the x-axis (the line $y=0$) and a curve $y=f(x)$, where the function $f(x)$ is continuous and non-negative over a certain interval $[a, b]$. This region, when revolved about the vertical line $x = c$, generates a three-dimensional solid.

Key Elements of the Problem:

- Region Boundaries: The region is enclosed between $y=0$ (the x-axis) and $y=f(x)$, with x-values ranging from a to b.
- Axis of Revolution: The line $x = c$, where c is a fixed constant, about which the region is rotated.
- Type of Solid: The rotation produces a solid of revolution, which can be visualized as a "washer" or a "shell" depending on the method used.

Understanding these elements helps determine the most suitable method for calculating the volume and provides geometric intuition about the shape of the resulting solid.

Choosing the Right Method: Disk/Washer vs. Shell

Calculating the volume of a solid of revolution can be approached via two primary methods:

1. Disk/Washer Method

Concept:
This method involves slicing the solid perpendicular to the axis of rotation, resulting in disks or washers whose volumes can be summed using integration.

Applicability:
- When the cross-sections perpendicular to the axis of rotation are disks or washers.
- When the region is bounded by simple curves, and the axis of rotation is parallel or perpendicular to the slices.

Visualization:
Imagine slicing the solid into thin disks stacked along the axis; the volume of each disk is π times the square of its radius times its thickness.

2. Shell Method

Concept:
This method involves slicing the solid into cylindrical shells parallel to the axis of revolution.

Applicability:
- When the region is better described in terms of horizontal or vertical strips that generate shells upon rotation.
- Particularly advantageous when the axis of rotation is not aligned with the axes of the slices.

Visualization:
Envision wrapping a thin rectangle around the axis of rotation, forming a shell; the volume of each shell is 2π times the radius times its height times its thickness.

Formulating the Problem: Mathematical Setup

Suppose the region R in the xy-plane is bounded by:

- The curve $y = f(x)$, where $f(x) \geq 0$ over $[a, b]$.
- The x-axis ($y=0$).

The goal: Find the volume of the solid obtained by revolving R about the vertical line $x = c$.

Assumptions and Notations:

- The function $f(x)$ is continuous on $[a, b]$.
- The line of revolution $x = c$ is fixed, with $c \in \mathbb{R}$.
- The interval $[a, b]$ defines the domain of the region.

Applying the Disk/Washer Method

Step 1: Visualize the Cross-Section

When rotating around $x=c$, the slices perpendicular to the x-axis are disks or washers, depending on whether the region is hollow or filled.

Step 2: Determine the Radius of the Disks

- For x in $[a, b]$, the distance from the axis of rotation $x=c$ to the slice at x is $|x - c|$.
- The radius of the disk at x is thus $|x - c|$.

Step 3: Write the Expression for the Volume Element

The volume of a thin disk at position x with thickness dx is:

$$ dV = \pi \times [\text{radius}]^2 \times dx = \pi \times (x - c)^2 \times dx $$

Note: The square ensures the radius is always positive, but since the radius is $|x - c|$, care must be taken to handle the absolute value.

Step 4: Set Up the Integral

The total volume V is obtained by integrating over the interval $[a, b]$:

$$
V = \int_a^b \pi \times (x - c)^2 \, dx
$$

If the region is bounded between $x=a$ and $x=b$, the integral becomes:

$$
V = \pi \int_a^b (x - c)^2 \, dx
$$

Step 5: Evaluate the Integral

The integral is straightforward:

$$
V = \pi \int_a^b (x^2 - 2cx + c^2) \, dx
$$

$$
V = \pi \left[\frac{x^3}{3} - c x^2 + c^2 x \right]_a^b
$$

Step 6: Final Expression

The volume is thus:

$$
V = \pi \left(\left[\frac{b^3}{3} - c b^2 + c^2 b \right] - \left[\frac{a^3}{3} - c a^2 + c^2 a \right] \right)
$$

This formula provides a direct way to calculate the volume when the region is bounded by the x-axis and the curve $y=f(x)$, with the axis of rotation at $x=c$.

Applying the Shell Method: An Alternative Approach

The shell method can sometimes simplify the process, especially when the region is described in terms of y, or when the axis of rotation is not aligned with the x-axis.

Step 1: Visualize the Shells

- Consider a vertical strip at x in [a, b], with width dx.
- Upon rotation about $x=c$, this strip generates a cylindrical shell.

Step 2: Determine the Shell Dimensions

- Radius: The distance from the strip at x to the line $x=c$, which is $|x - c|$.
- Height: The height of the strip is $y = f(x)$.

Step 3: Write the Shell Volume Element

$$
dV = 2\pi \times (\text{radius}) \times (\text{height}) \times dx = 2\pi |x - c| f(x) \, dx
$$

Step 4: Set Up the Integral

$$
V = \int_a^b 2\pi |x - c| f(x) \, dx
$$

The absolute value ensures correctness depending on whether $x > c$ or $x < c$.

Step 5: Handling the Absolute Value

- Split the integral at $x=c$:

$$
V = 2\pi \left(\int_a^c (c - x) f(x) \, dx + \int_c^b (x - c) f(x) \, dx \right)
$$

- Note that $(c - x)$ is positive when $x < c$, and $(x - c)$ is positive when $x > c$.

Step 6: Final Expression

The total volume is:

$$
V = 2\pi \left(\int_a^c (c - x) f(x) \, dx + \int_c^b (x - c) f(x) \, dx \right)
$$

This approach is particularly advantageous when the function $f(x)$ is complex or when the region extends across the axis of rotation.

Special Cases and Geometric Insights

The general formulas adapt to various specific scenarios:

1. Rotation About $x = c$ with Symmetric Regions

If the region is symmetric about $x=c$, the calculations can be simplified by exploiting symmetry.

2. When the Region is Bounded by a Function and the X-axis

If the region is between $y=0$ and $y=f(x)$, and $f(x)$ is known explicitly, the volume can be expressed as a definite integral involving $f(x)$.

3. Rotation About $x = c$ Outside the Region

- When $c < a$ or $c > b$, the formulas hold, but the absolute value in the shell method becomes critical.

4. Volume of a Sphere or Cylinder

Special cases where $f(x)$ describes a circle or a rectangle lead to well-known volume formulas for spheres or cylinders, acting as validation checks.

Examples to Illustrate the Concepts

Example 1: Rotating a Semicircular Region

Suppose the region bounded by $y = \sqrt{r^2 - (x - h)^2}$, x in $[h - r, h + r]$, is rotated about $x = c$.

- The resulting solid is a sphere of radius r centered at $(h, 0)$.
- The

Find The Volume Of The Solid Obtained By Rotating About The X Line The Region Bounded By The X Axis

Find The Volume Of The Solid Obtained By Rotating About The X Line The Region Bounded By The X Axis

Related Articles

- Complete The Photosynthesis Reaction By Placing The Compounds And Energy Sources Into The Reaction As
- Explain ONE Way Islamic Culture Blended With Other Cultures In Al-Andalus During The Period C. 1200c.
- Caroline Works For A Corporation In Oriel, A City That Accommodates More Than 10 Million Inhabitants.

Back to Home