Find Three Positive Numbers, The Sum Of Which Is 51 , So That The Sum Of Their Squares Is As Small As

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Mathematics often challenges us to find optimal solutions under given constraints. One intriguing problem involves discovering three positive numbers whose sum equals a specific value—in this case, 51—while minimizing the sum of their squares. This type of problem combines elements of algebra, calculus, and optimization, providing valuable insights into how constraints influence optimal solutions.

Understanding how to identify such numbers not only enhances problem-solving skills but also deepens comprehension of quadratic functions, derivatives, and the principles of optimization. Whether you're a student preparing for exams, a teacher designing engaging problems, or a math enthusiast interested in elegant solutions, this article offers a comprehensive exploration of the problem, its mathematical foundation, and step-by-step methods to arrive at the optimal numbers.

Understanding the Problem

The problem can be formalized as follows:

- Let the three positive numbers be x, y, and z.
- The sum constraint: $x + y + z = 51$.
- The objective: Minimize the sum of their squares, $S = x^2 + y^2 + z^2$.

The challenge is to find the values of x, y, and z that satisfy the sum constraint while producing the smallest possible value for S.

Why is this problem interesting?
It combines the linear constraint (sum equals 51) with a quadratic objective function (sum of squares). Since the sum of squares grows rapidly with larger numbers, intuitively, the optimal solution balances the numbers to be as close as possible to each other, minimizing the sum of their squares.

Mathematical Foundations

Before solving the problem, it's important to understand some key concepts:

Quadratic Functions and Convexity

The function $f(x) = x^2$ is convex, meaning that the line segment between any two points on the parabola lies above the curve. When summing quadratic functions, the overall function remains convex, which ensures the existence of a unique global minimum.

Optimization with Constraints

The problem involves a constraint $x + y + z = 51$, which can be addressed using methods like substitution, Lagrange multipliers, or symmetry considerations.

Approach to the Solution

The key idea is that, under a fixed sum, the sum of squares is minimized when the numbers are as equal as possible. This is grounded in the principle that the quadratic function penalizes larger deviations, so evenly distributing the total sum among the numbers minimizes the total sum of their squares.

Step-by-step plan:

1. Assumption of equality:
Since the problem is symmetric in x, y, and z, the minimal sum of squares occurs when all three numbers are equal, i.e., $x = y = z$.

2. Calculate the equal values:
Given the sum $x + y + z = 51$, and assuming equality,
$$
3x = 51 \Rightarrow x = y = z = \frac{51}{3} = 17.
$$

3. Verify the solution:
Substituting back,
$$
S = 3 \times 17^2 = 3 \times 289 = 867.
$$

This suggests that the three numbers are all 17, and the sum of their squares is 867.

Validation of the Solution Using Calculus

While the symmetry argument strongly indicates the optimal solution, it's prudent to verify this using calculus, specifically the method of Lagrange multipliers, which handles constrained optimization problems elegantly.

Applying Lagrange Multipliers

- Objective function:
$$
S(x, y, z) = x^2 + y^2 + z^2
$$

- Constraint:
$$
g(x, y, z) = x + y + z - 51 = 0
$$

- Lagrangian function:
$$
\mathcal{L}(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda (x + y + z - 51)
$$

- Partial derivatives:
$$
\frac{\partial \mathcal{L}}{\partial x} = 2x - \lambda = 0 \Rightarrow 2x = \lambda
$$
$$
\frac{\partial \mathcal{L}}{\partial y} = 2y - \lambda = 0 \Rightarrow 2y = \lambda
$$
$$
\frac{\partial \mathcal{L}}{\partial z} = 2z - \lambda = 0 \Rightarrow 2z = \lambda
$$

- Solution:
Since all three derivatives equal zero,
$$
2x = 2y = 2z \Rightarrow x = y = z
$$

- Using the constraint:
$$
3x = 51 \Rightarrow x = y = z = 17
$$

This confirms that the minimal sum of squares occurs when all three numbers are equal to 17.

Alternative Scenarios and Constraints

While the above solution assumes positive numbers and equality, it’s instructive to consider variations:

- What if the numbers are not all positive?
The problem states "positive numbers," so $x, y, z > 0$. The solution $x = y = z = 17$ satisfies this condition.

- What if the sum is different?
The method generalizes: for any total sum T, the minimal sum of squares occurs when each number equals $T/3$.

- What if the numbers are constrained further?
For instance, if each number must be at least 10, then the equal distribution might not be feasible, and a different approach would be necessary.

Practical Applications of the Problem

Understanding how to minimize the sum of squares under a sum constraint has numerous real-world applications:

1. Resource Allocation:
Distributing resources evenly to minimize variance or inefficiency.

2. Data Science and Statistics:
Minimizing the sum of squared errors to find the best fit in regression analysis.

3. Engineering Design:
Balancing loads or stresses across components for optimal performance.

4. Financial Portfolio Management:
Distributing investments evenly to minimize overall risk.

Summary and Key Takeaways

- When seeking three positive numbers with a fixed sum to minimize the sum of their squares, the optimal solution is to distribute the total evenly.

- For the sum $x + y + z = 51$, the three numbers are each 17.

- The minimal sum of squares at this point is 867.

- The approach combines symmetry, calculus (Lagrange multipliers), and fundamental principles of convex functions.

Conclusion

Finding the three positive numbers that sum to 51 while minimizing the sum of their squares is a classic problem illustrating the power of symmetry and calculus in optimization. By understanding the underlying principles, you can quickly determine that evenly distributing the total sum among the numbers yields the smallest possible sum of their squares.

This problem exemplifies how mathematical concepts are interconnected and applicable across various fields. Whether in theoretical mathematics, applied sciences, or everyday decision-making, such optimization techniques are invaluable tools for achieving the best possible outcomes under given constraints.

Additional Resources for Further Learning

- Calculus and Optimization Textbooks:
Explore chapters on constrained optimization and Lagrange multipliers for a deeper understanding.

- Mathematical Problem Solving Websites:
Platforms like Brilliant.org or Khan Academy offer interactive problems and tutorials on similar topics.

- Software Tools:
Use graphing calculators or software like Wolfram Alpha, GeoGebra, or Desmos to visualize functions and constraints.

Remember:
The key to solving similar problems lies in recognizing symmetry, applying calculus techniques, and understanding the nature of quadratic functions. With practice, these methods become powerful tools for tackling a wide array of optimization problems.

Frequently Asked Questions

How do I find three positive numbers whose sum is 51 and whose sum of squares is minimized?

You can set up the problem using variables for the three numbers, say x, y, and z, with the constraint $x + y + z = 51$. To minimize the sum of their squares, apply methods like Lagrange multipliers or symmetry to find that the numbers should be equal, i.e., each is 17, resulting in the minimal sum of squares.

Why are the three numbers all equal when trying to minimize their sum of squares given a fixed sum?

Because of the symmetry and the convexity of the quadratic function, the minimum sum of squares occurs when the three numbers are equal. In this case, each number is 17, since 51 divided by 3 equals 17.

What is the minimal sum of the squares for the three numbers summing to 51?

If each number is 17, then the sum of their squares is $17^2 + 17^2 + 17^2 = 3 \times 289 = 867$, which is the minimum possible.

Can this problem be solved using calculus or algebraic methods?

Yes, by setting up the problem with a constraint and applying calculus techniques such as Lagrange multipliers, or through algebraic reasoning, you can determine the numbers that minimize the sum of squares.

What is the general principle behind minimizing the sum of squares given a fixed sum?

The principle is that, for positive numbers with a fixed total sum, the sum of their squares is minimized when all the numbers are equal, due to the convexity of the quadratic function.

Are there other approaches besides calculus to solve this problem?

Yes, you can use symmetry arguments, optimization principles, or algebraic methods like completing the square, but calculus provides a straightforward and rigorous approach to find the minimum.

Additional Resources

Find Three Positive Numbers, The Sum Of Which Is 51, So That The Sum Of Their Squares Is As Small As — this intriguing problem blends elements of algebra, calculus, and optimization. It challenges us to discover three positive numbers whose total adds up to a fixed value (51), while simultaneously minimizing the sum of their squares. This type of problem is common in mathematical optimization, illustrating how constraints shape solutions and how calculus can be used to find optimal points. Whether you're a student tackling calculus problems, a teacher preparing lesson materials, or a professional interested in optimization techniques, understanding how to approach this problem offers valuable insights into mathematical reasoning.

Understanding the Problem

Before diving into the solution, it’s essential to clearly understand the problem's statement:

- Given: Three positive numbers, say x, y, and z
- Constraint: Their sum must be 51, i.e., $x + y + z = 51$
- Objective: Minimize the sum of their squares, i.e., $x^2 + y^2 + z^2$

This is a classic constrained optimization problem, which we can approach through methods like Lagrange multipliers or by leveraging symmetry and algebraic techniques.

Breaking Down the Problem

Why Minimize the Sum of Squares?

The sum of squares $S = x^2 + y^2 + z^2$ is a measure of the spread or variance of the numbers. Intuitively, for a fixed total sum, the sum of squares is minimized when the numbers are as close to each other as possible. This is because larger disparities tend to inflate the sum of squares, owing to the quadratic growth.

Key Insight

Given the symmetry of the problem (no number is favored over the others), the minimal sum of squares, under the fixed sum constraint, occurs when the three numbers are equal—if the positivity constraint is satisfied.

Step-by-Step Approach

Step 1: Formulate the Problem Mathematically

- Variables: $x, y, z > 0$
- Constraint: $x + y + z = 51$

- Minimize: $S = x^2 + y^2 + z^2$

Step 2: Use Symmetry and Expectation

Since the problem is symmetric in x, y, z, the minimal sum of squares often occurs when they are equal, provided this satisfies the constraints.

- Hypothesis: $x = y = z$

- Applying the constraint: $3x = 51 \Rightarrow x = 17$

- Check positivity: All numbers are positive (17), so this is valid.

- Compute the sum of squares: $3 \times 17^2 = 3 \times 289 = 867$

This gives us a candidate solution: $(17, 17, 17)$ with total sum 51 and sum of squares 867.

Verifying Optimality

While the symmetric solution seems promising, let's verify that it indeed minimizes the sum of squares:

- Intuitive reasoning: For a fixed sum, the sum of squares is minimized when the numbers are equal—this can be proven using the Cauchy-Schwarz inequality or Jensen's inequality.

- Formal proof (via calculus):

Suppose we fix $x + y + z = 51$, with $x, y, z > 0$. Let's attempt to minimize $S = x^2 + y^2 + z^2$.

Using Lagrange multipliers:

- Set up the Lagrangian:

$$
\mathcal{L} = x^2 + y^2 + z^2 - \lambda (x + y + z - 51)
$$

- Take partial derivatives:

$$
\frac{\partial \mathcal{L}}{\partial x} = 2x - \lambda = 0 \Rightarrow 2x = \lambda
$$

$$
\frac{\partial \mathcal{L}}{\partial y} = 2y - \lambda = 0 \Rightarrow 2y = \lambda
$$

$$
\frac{\partial \mathcal{L}}{\partial z} = 2z - \lambda = 0 \Rightarrow 2z = \lambda
$$

- From these, it follows that:

$$
2x = 2y = 2z \Rightarrow x = y = z
$$

- Applying the sum constraint:

$$
3x = 51 \Rightarrow x = y = z = 17
$$

Thus, the critical point occurs at $x = y = z = 17$.

- Check positivity: All are positive, confirming the candidate is valid.

Final Answer and Interpretation

The three positive numbers, the sum of which is 51, so that the sum of their squares is as small as possible, are all equal to 17. The minimal sum of squares in this case is:

$$
\boxed{
\text{Minimum sum of squares} = 3 \times 17^2 = 867
}
$$

Broader Applications and Insights

This problem exemplifies a fundamental principle in optimization: for a fixed sum, the sum of squares is minimized when all numbers are equal. This principle extends beyond algebra into various fields:

- Economics: Distributing resources evenly minimizes variance.
- Statistics: Equal variance across groups minimizes overall variability.
- Engineering: Balancing loads across components reduces stress.

Variations and Extensions

While the current problem deals with three positive numbers, similar principles apply to:

- Different numbers of variables (e.g., four, five, or more).
- Constraints involving inequalities (e.g., some numbers must be greater than a certain

value).
- Minimizing other functions, such as the sum of absolute values or higher powers.

Conclusion

Finding three positive numbers whose sum is 51 and minimizing the sum of their squares is a classic optimization problem that demonstrates the power of symmetry and calculus. The key takeaway is that equal distribution yields the optimal solution under these conditions. This problem not only sharpens problem-solving skills but also provides foundational insights applicable across various scientific and mathematical disciplines.

Summary Checklist

- Problem understanding: Fixed sum, minimize sum of squares.
- Key insight: Equal numbers minimize the sum of squares.
- Solution: \(x = y = z = 17 \).
- Minimum sum of squares: 867.
- Broader significance: Principles of symmetry, optimization, and resource distribution.

If you want to explore further, consider how these techniques apply to more complex constraints or higher dimensions. The principles of symmetry and calculus are powerful tools in the quest for optimal solutions across many fields.

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