For The Fourier Series Given Below, Deduce A Series For At The Point X = 5. Use The Sigma Notation To

For The Fourier Series Given Below, Deduce A Series For At The Point X = 5. Use The Sigma Notation To explore the process of deriving a series representation for a specific function at a given point, focusing on Fourier series expansion and sigma notation. Fourier series are a powerful tool in mathematical analysis, enabling the representation of periodic functions as infinite sums of sines and cosines. Understanding how to manipulate and evaluate these series at particular points is essential in fields ranging from signal processing to heat transfer. This article provides a comprehensive guide to deducing the series for a function at $x = 5$, illustrating the process step-by-step with clear explanations and examples.

Understanding Fourier Series: An Overview

What Is a Fourier Series?

A Fourier series decomposes a periodic function $f(x)$ into an infinite sum of sines and cosines. It is expressed as:

$$
f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)
$$

where:
- a_0 is the average or mean value of the function over one period.
- a_n and b_n are the Fourier coefficients, determining the amplitude of each harmonic.

Significance of Fourier Series

Fourier series allow us to:
- Approximate complex periodic functions with simpler trigonometric functions.
- Analyze signals in engineering.
- Solve differential equations with periodic boundary conditions.
- Study heat conduction, wave propagation, and acoustics.

Given Fourier Series and Objective

Suppose we are provided with a specific Fourier series expansion of a function $f(x)$:

$$
f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)
$$

Our goal is to find the value of A at $x = 5$, where A is expressed as a series in sigma notation, possibly involving the Fourier coefficients.

Key Points:
- The function $f(x)$ is periodic with period $2L$.
- The series coefficients a_n and b_n are typically calculated from integrals over one period.
- We need to evaluate the series at $x=5$, which involves substituting $x=5$ into the series and simplifying.

Step-by-Step Deduction of the Series at $x = 5$

Step 1: Write Down the Fourier Series Expression

Start with the general form:

$$
f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)
$$

For the specific problem, identify a_0, a_n, and b_n.

Step 2: Calculate Fourier Coefficients

The coefficients are given by:

$$
a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx
$$

$$
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx
$$

$$
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx
$$

These integrals are computed over one period.

Step 3: Substituting $x=5$ into the Series

Once the coefficients are known, substitute $x=5$:

$$
A = f(5) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi \times 5}{L} + b_n \sin \frac{n\pi \times 5}{L} \right)
$$

This expresses A as an infinite series involving sigma notation.

Step 4: Expressing the Series in Sigma Notation

The series can be written as:

$$
A = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \left(\frac{n\pi \times 5}{L} \right) + b_n \sin \left(\frac{n\pi \times 5}{L} \right) \right)
$$

which is concise and suitable for further analysis or computational evaluation.

Using Sigma Notation for Simplification

Unified Series Representation

Often, it is advantageous to combine the cosine and sine terms into a single sigma notation using phase shifts or complex exponentials. For example:

$$
f(x) = \sum_{n=1}^{\infty} c_n \cos \left(\frac{n\pi x}{L} - \phi_n \right)
$$

where c_n and ϕ_n are amplitude and phase, respectively.

Advantages of Sigma Notation

- Compact representation of infinite sums.
- Easier to analyze convergence.
- Facilitates numerical computation.
- Simplifies algebraic manipulations when evaluating at specific points.

Application and Examples

Example 1: Square Wave Function

Suppose $f(x)$ is a square wave with period $2L$. Its Fourier series contains only sine terms:

$$
f(x) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^\infty \frac{1}{n} \sin \frac{n\pi x}{L}
$$

To find the series at $x=5$:

$$
A = \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^\infty \frac{1}{n} \sin \frac{n\pi \times 5}{L}
$$

Expressed in sigma notation, this becomes:

$$
A = \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^\infty \frac{1}{n} \sin \left(\frac{n\pi \times 5}{L} \right)
$$

Example 2: Cosine Series for Even Functions

If $f(x)$ is an even function, the Fourier series simplifies to only cosine terms:

$$
f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}
$$

Evaluating at $x=5$:

$$
A = a_0 + \sum_{n=1}^{\infty} a_n \cos \left(\frac{n\pi \times 5}{L} \right)
$$

Summary and Practical Considerations

- The process involves calculating Fourier coefficients from the original function.
- Once coefficients are known, substituting $x=5$ provides the value A as an infinite sum.
- Expressing the series using sigma notation enhances clarity and computational efficiency.
- Numerical approximation involves truncating the series after a finite number of terms, with the error depending on the function's properties.

Conclusion: From Series to Value at a Point

Deducting a series for A at $x=5$ from a Fourier series involves understanding the structure of the Fourier expansion, calculating the coefficients, and then substituting the specific point into the series. Using sigma notation streamlines the representation, making it easier to analyze, approximate, and understand the behavior of the function at that point. Mastery of these techniques is fundamental in mathematical analysis, signal processing, and many engineering disciplines, providing a robust framework for analyzing periodic functions in both theoretical and applied contexts.

Frequently Asked Questions

How do I determine the Fourier series for a function at a specific point, such as x = 5?

To find the Fourier series at $x = 5$, first derive the general Fourier series expression for the function, then evaluate it at $x = 5$ by substituting the value into the series. Using sigma notation, this involves summing the Fourier coefficients multiplied by the corresponding sine or cosine functions at $x = 5$.

What is the process to express a Fourier series at x = 5 using sigma notation?

The process involves identifying the Fourier coefficients (a_n and b_n), writing the Fourier series in sigma notation as a sum over n, and then substituting $x = 5$ into the series. This results in an expression like: $f(5) = a_0/2 + \Sigma (a_n \cos(n\pi 5/L) + b_n \sin(n\pi 5/L))$, where L is the period parameter.

How can I simplify the Fourier series at a specific point using sigma notation?

Simplification involves calculating the Fourier coefficients and plugging in $x = 5$. In sigma notation, each term becomes a known numeric value, allowing you to write the series as a sum of constants multiplied by sine or cosine functions evaluated at $x = 5$. This often reduces to a manageable finite or infinite sum depending on the coefficients.

Are there any common pitfalls when deducing the Fourier series at x = 5 using sigma notation?

Yes, common pitfalls include incorrect computation of Fourier coefficients, misapplication of the sigma notation (such as wrong limits or missing terms), and errors in substituting $x = 5$ into the series. Ensuring accurate coefficient calculation and careful substitution helps avoid these mistakes.

What are the key steps to find the series At x = 5 from a given Fourier series using sigma notation?

Key steps include: (1) Write down the general Fourier series in sigma notation; (2) Calculate or obtain the Fourier coefficients; (3) Substitute x

= 5 into the series; (4) Simplify each term accordingly; (5) Sum the series to find the value of At x = 5, which may involve recognizing patterns or convergence considerations.

Additional Resources

For The Fourier Series Given Below, Deduce A Series For At The Point X = 5. Use The Sigma Notation To

---

Introduction

Fourier series serve as a cornerstone in mathematical analysis, providing a powerful tool for representing periodic functions as an infinite sum of sines and cosines. Their utility spans numerous disciplines, including engineering, physics, signal processing, and applied mathematics. The ability to extract specific function values from a Fourier series, such as the value of a function at a particular point, is fundamental for both theoretical insights and practical applications.

This article embarks on a comprehensive investigation into deducing a series representation for the value of a function $A(t)$ at the point $t = 5$, given its Fourier series expansion. We will delve into the theoretical underpinnings of Fourier series, explore the methodology for isolating the desired value, and articulate the process using sigma notation—an essential aspect for clarity and formal mathematical expression.

Background on Fourier Series

Fundamentals of Fourier Series

Fourier series decompose a periodic function $f(t)$ with period T into an infinite sum of sines and cosines:

$$
f(t) = a_0 + \sum_{n=1}^\infty \left(a_n \cos \frac{2\pi n t}{T} + b_n \sin \frac{2\pi n t}{T} \right)
$$

where the coefficients a_n and b_n are determined via integrals over one period:

$$
a_n = \frac{2}{T} \int_0^T f(t) \cos \frac{2\pi n t}{T} dt, \quad
b_n = \frac{2}{T} \int_0^T f(t) \sin \frac{2\pi n t}{T} dt
$$

The constant term a_0 is:

$$
a_0 = \frac{1}{T} \int_0^T f(t) dt
$$

Convergence and Pointwise Evaluation

Under certain conditions (e.g., piecewise smoothness), Fourier series converge to the original function at all points, except possibly at points of discontinuity where it converges to the average of the left and right limits. To evaluate $A(5)$, we typically leverage the series’ convergence properties, provided the series converges uniformly or pointwise at $t=5$.

The Core Problem: Deducing $A(5)$ From Its Fourier Series

Suppose the Fourier series of $A(t)$ is given (or can be derived), and our goal is to express $A(5)$ explicitly, using a sigma notation that encapsulates the infinite sum of terms involving the Fourier coefficients.

This problem involves several key steps:

1. Understanding the Fourier series expansion for $A(t)$,
2. Recognizing the form of the series at $t=5$,
3. Expressing the series in sigma notation, explicitly including the evaluation point,
4. Simplifying or computing the series to obtain a numerical or functional expression for $A(5)$.

Methodology

Step 1: Fourier Series Representation of $A(t)$

Assuming $A(t)$ is periodic with period T, its Fourier series is:

$$
A(t) = a_0 + \sum_{n=1}^\infty \left(a_n \cos \frac{2\pi n t}{T} + b_n \sin \frac{2\pi n t}{T} \right)
$$

The coefficients a_n, b_n are known or can be calculated via integrals.

Step 2: Evaluation at $t=5$

Substituting $t=5$:

\[

$$
A(5) = a_0 + \sum_{n=1}^\infty \left(a_n \cos \frac{2\pi n \times 5}{T} + b_n \sin \frac{2\pi n \times 5}{T} \right)
$$

This sum converges under suitable conditions, and the value of $A(5)$ can be expressed as an infinite series.

Step 3: Expressing the Series Using Sigma Notation

The series can be rewritten explicitly as:

$$
A(5) = a_0 + \sum_{n=1}^\infty \left[a_n \cos \left(\frac{10 \pi n}{T} \right) + b_n \sin \left(\frac{10 \pi n}{T} \right) \right]
$$

This is the explicit sigma notation form, capturing all terms in the infinite sum at the specific point.

Deep Dive: Formal Sigma Notation and Its Derivation

General Series Expression

The Fourier series at $t=5$:

$$
A(5) = a_0 + \sum_{n=1}^\infty \left(a_n \cos \theta_n + b_n \sin \theta_n \right)
$$

where

$$
\theta_n = \frac{2 \pi n \times 5}{T}
$$

Putting this in a more compact form:

$$
A(5) = a_0 + \sum_{n=1}^\infty \left(a_n \cos \theta_n + b_n \sin \theta_n \right)
$$

This expression allows us to evaluate $A(5)$ directly once the Fourier coefficients are known.

Explicit Coefficient Calculations

Suppose the Fourier coefficients are known or can be derived from the original function $A(t)$. For example:

$$
a_n = \frac{2}{T} \int_{0}^{T} A(t) \cos \left(\frac{2 \pi n t}{T} \right) dt
$$
$$
b_n = \frac{2}{T} \int_{0}^{T} A(t) \sin \left(\frac{2 \pi n t}{T} \right) dt
$$

Once these are obtained, the entire series at $t=5$ can be written in sigma notation as:

$$
A(5) = a_0 + \sum_{n=1}^\infty \left[\left(\frac{2}{T} \int_{0}^{T} A(t) \cos \frac{2 \pi n t}{T} dt \right) \cos \left(\frac{10 \pi n}{T} \right) + \left(\frac{2}{T} \int_{0}^{T} A(t) \sin \frac{2 \pi n t}{T} dt \right) \sin \left(\frac{10 \pi n}{T} \right) \right]
$$

This expression encapsulates the entire process: the Fourier coefficients are integrals over the period, and the evaluation at $t=5$ involves plugging in the specific angle θ_n.

Practical Considerations and Examples

Example: Square Wave Function

Suppose $A(t)$ is a square wave with period $T=2$, and amplitude 1 within $[0,1)$, -1 within $[1,2)$.

- The Fourier series of this square wave is well-known:

$$
A(t) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^\infty \frac{1}{n} \sin \left(\frac{\pi n t}{T/2} \right)
$$

- To find $A(5)$, we substitute $t=5$:

$$
A(5) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^\infty \frac{1}{n} \sin \left(\pi n \times \frac{5}{1} \right)
$$

- Recognizing that $\sin (\pi n \times 5) = \sin (5 \pi n)$, and since $\sin(k \pi n) = 0$ for integer k, n, the sum simplifies, or the series collapses, depending on the parity.

This demonstrates how the series evaluation reduces to known trigonometric identities and the sigma notation explicitly encodes the sum.

Significance and Broader Implications

Analytical Utility

Expressing $A(5)$ via the Fourier series in sigma notation allows for:

- Precise numerical computation, especially when the Fourier coefficients are explicitly known.
- Analytical insights into how the function's periodic components influence its value at specific points.
- The capacity to handle functions with discontinuities or complex behavior through convergence properties.

Applications in Signal Processing

In signal processing, reconstructing a signal's value at a specific instant from its Fourier expansion is essential for filtering, sampling, and analysis. The sigma notation formalizes this process, enabling algorithmic implementation.

Limitations and Challenges

- Convergence issues: The Fourier series may converge slowly or exhibit Gibbs phenomena near discontinuities, affecting the accuracy at $t=5$.
- Coefficient Calculation Complexity: For complex functions, computing Fourier coefficients involves challenging integrals.
- Periodicity Constraints: The method assumes the function is periodic; non-periodic functions require Fourier transforms or other techniques.

Conclusion

[For The Fourier Series Given Below Deduce A Series For At The Point X 5 Use The Sigma Notation To]()

For The Fourier Series Given Below Deduce A Series For At The Point X 5 Use The Sigma Notation To

Related Articles

- Four Wires (red,green, Blue And Yellow) Need To Be Attached To A Circuit Board. A Robotic Device Will
- If The Daily Marginal Cost For A Product Is Mc = 6x 140, With Fixed Costs Amounting To $300, Find The
- How Does Epigenetics (from Bio, Mod 14) Explain How Genes And The Environment Influence Intelligence?

Back to Home