Given Field F = X Ax + Y Ay. Evaluate The Left Side Of Green's Theorem Where C -> Rectangular Path

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Introduction

Green's Theorem is a fundamental result in vector calculus that relates a line integral around a simple closed curve C to a double integral over the region D enclosed by C. It provides a powerful tool for converting complex line integrals into easier double integrals and vice versa. In this article, we explore how to evaluate the left side of Green's Theorem for a specific vector field $\mathbf{F} = x \mathbf{A}_x + y \mathbf{A}_y$, where the curve C is a rectangular path. This derivation involves understanding the structure of the vector field, setting up the line integral, and simplifying it accordingly.

Understanding the Vector Field $\mathbf{F} = x \mathbf{A}_x + y \mathbf{A}_y$

The Components of the Field $\mathbf{F}$

The vector field $\mathbf{F}$ is given as:

$$
\mathbf{F} = x \mathbf{A}_x + y \mathbf{A}_y
$$

Here,

- x and y are the Cartesian coordinates.
- $\mathbf{A}_x$ and $\mathbf{A}_y$ are the unit vectors in the x- and y-directions, respectively.

This is a simple, linear vector field that depends directly on the spatial coordinates. Such fields often model physical phenomena like velocity fields in fluid flow or force fields in physics.

Visualizing the Field

- The x-component of $\mathbf{F}$ grows linearly along the x-axis.
- The y-component grows linearly along the y-axis.
- The field points radially outward from the origin, with magnitude increasing with distance from the origin.

Setting Up Green's Theorem

Green's Theorem Statement

Green's Theorem states:

$$
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA
$$

where:

- $\mathbf{F} = P \mathbf{A}_x + Q \mathbf{A}_y$,
- C is a positively oriented, simple closed curve,
- D is the region enclosed by C.

Applying to Our Field

Given $P = x$ and $Q = y$, the line integral's left side becomes:

$$
\oint_C P\, dx + Q\, dy = \oint_C x\, dx + y\, dy
$$

Our task is to evaluate this line integral over the rectangle C.

Defining the Rectangular Path C

Coordinates of the Rectangle

Suppose the rectangle C is bounded by:

- x from a to b,
- y from c to d.

The boundary C consists of four segments:

1. Segment 1: from (a, c) to (b, c) (bottom side)
2. Segment 2: from (b, c) to (b, d) (right side)
3. Segment 3: from (b, d) to (a, d) (top side)
4. Segment 4: from (a, d) to (a, c) (left side)

Evaluating the Line Integral Over the Rectangle

Step 1: Parameterize Each Segment

For each segment, define $x(t)$ and $y(t)$, compute dx and dy, and set up the integrals.

Segment 1: Bottom side

- x from a to b,
- $y = c$,
- t in $[a, b]$,

$$
x = t, \quad y = c, \quad dx = dt, \quad dy=0
$$

Line integral:

$$
I_1 = \int_{a}^{b} x\, dx + y\, dy = \int_{a}^{b} t\, dt + c \times 0 = \int_{a}^{b} t\, dt
$$

Segment 2: Right side

- $x = b$,
- y from c to d,

$$
x = b, \quad y = t, \quad dx=0, \quad dy=dt
$$

Integral:

$$
I_2 = \int_{c}^{d} x\, dx + y\, dy = \int_{c}^{d} b \times 0 + t\, dt = \int_{c}^{d} t\, dt
$$

Segment 3: Top side

- x from b to a,
- $y = d$,

$$
x = t, \quad y = d, \quad dx=dt, \quad dy=0
$$

Integral:

$$
I_3 = \int_{b}^{a} t\, dt + d \times 0 = \int_{b}^{a} t\, dt
$$

Segment 4: Left side

- $x = a$,
- y from d to c,

$$
x = a, \quad y = t, \quad dx=0, \quad dy=dt
$$

Integral:

$$
I_4 = \int_{d}^{c} a \times 0 + t\, dt = \int_{d}^{c} t\, dt
$$

Step 2: Calculate Each Segment's Integral

Remember to account for orientation; the path is traversed counterclockwise, so the limits for segments 3 and 4 are in reverse.

Segment 1:

$$
I_1 = \left[\frac{t^2}{2} \right]_a^b = \frac{b^2}{2} - \frac{a^2}{2}
$$

Segment 2:

$$
I_2 = \left[\frac{t^2}{2} \right]_c^d = \frac{d^2}{2} - \frac{c^2}{2}
$$

Segment 3:

$$
I_3 = \left[\frac{t^2}{2} \right]_b^a = \frac{a^2}{2} - \frac{b^2}{2}
$$

Segment 4:

$$
I_4 = \left[\frac{t^2}{2} \right]_d^c = \frac{c^2}{2} - \frac{d^2}{2}
$$

Step 3: Sum the Integrals

Total line integral:

\[

\oint_C x\, dx + y\, dy = I_1 + I_2 + I_3 + I_4
\]

Substituting:

\[
= \left(\frac{b^2}{2} - \frac{a^2}{2} \right) + \left(\frac{d^2}{2} - \frac{c^2}{2} \right) + \left(\frac{a^2}{2} - \frac{b^2}{2} \right) + \left(\frac{c^2}{2} - \frac{d^2}{2} \right)
\]

Simplify:

\[
= \left(\frac{b^2}{2} - \frac{a^2}{2} + \frac{a^2}{2} - \frac{b^2}{2} \right) + \left(\frac{d^2}{2} - \frac{c^2}{2} + \frac{c^2}{2} - \frac{d^2}{2} \right) = 0
\]

Result: The line integral around the rectangle is zero.

Interpretation and Implications

Why is the line integral zero?

The calculation shows that for the vector field \(\mathbf{F} = x \mathbf{A}_x + y \mathbf{A}_y \), the circulation around a rectangle aligned with the axes cancels out completely. This is consistent with the fact that the curl of \(\mathbf{F} \) in this case is zero, indicating the field is irrotational over this domain.

Connection to Green's Theorem

Green's Theorem states:

\[
\oint_C P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA
\]

Calculate the curl:

\[
\frac{\partial Q}{\partial x} = 0, \quad \frac{\

Frequently Asked Questions

What is the main objective when evaluating the left side of Green's Theorem for the given field F = X Ax + Y Ay over a rectangular path?

The main objective is to compute the line integral of the vector field F around the rectangular path and verify it against the double integral of the curl of F over the enclosed region, as per Green's Theorem.

How do you parameterize the rectangular path C for evaluating the line integral in Green's Theorem?

You parameterize each side of the rectangle using straight-line segments, typically defining points $(x1, y1)$, $(x2, y2)$, etc., and then integrate $F \cdot dr$ along each segment, ensuring proper orientation (counterclockwise).

What is the curl of the given field F = X Ax + Y Ay, and how is it used in Green's Theorem?

The curl of F in 2D is $(\partial Y/\partial x - \partial X/\partial y)$. For F = X Ax + Y Ay, this becomes $(\partial Y/\partial x - \partial X/\partial y) = 0 - 0 = 0$, indicating the field is irrotational, which simplifies the evaluation.

How does the rectangular path's orientation affect the evaluation of the line integral in Green's Theorem?

The path should be oriented counterclockwise; reversing the orientation changes the sign of the line integral, so maintaining consistent orientation is crucial for accurate evaluation.

What is the significance of evaluating the left side of Green's Theorem in the context of vector calculus?

Evaluating the left side (line integral) helps verify the relationship between circulation around a closed curve and the flux of curl through the enclosed area, illustrating the fundamental theorem relating line and double integrals.

In the context of a rectangular path, how do you compute the line integral of F = X Ax + Y Ay along each side?

You substitute the parametric equations for each side into F, compute $F \cdot dr$, and integrate over the respective segment, summing the results for all four sides to find the total line integral.

Given the field F = X Ax + Y Ay and a rectangular path with vertices at (a, c), (b, c), (b, d), (a, d), how would you proceed to evaluate the left side of Green's Theorem?

You parametrize each side between these vertices, compute the line integral of F along each, sum the four integrals, ensuring proper orientation, to find the total circulation around the rectangle.

Additional Resources

Given Field F = X Ax + Y Ay. Evaluate The Left Side Of Green's Theorem Where C → Rectangular Path

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Introduction

In the realm of vector calculus, Green's theorem serves as a cornerstone for connecting the circulation around a closed curve to the flux of a vector field through the enclosed region. At its core, it translates a line integral around a simple, closed curve into a double integral over the region it encloses, providing a powerful tool for evaluating complex integrals with relative ease. When faced with a vector field such as F = X Ax + Y Ay, understanding how to evaluate the left side of Green's theorem—specifically, the line integral around a rectangular path—becomes essential for both theoretical insights and practical applications across physics, engineering, and mathematics.

This article aims to thoroughly analyze this problem: starting with the foundational concepts, delving into the specifics of the given vector field, calculating the line integral along a rectangular path, and finally, interpreting the results within the framework of Green's theorem. We will explore each aspect with detailed explanations, illustrative examples, and critical insights to provide a comprehensive understanding.

Background: Green’s Theorem and Its Significance

What Is Green’s Theorem?

Green’s theorem is a fundamental relation in vector calculus that connects a line integral around a simple, closed curve C to a double integral over the region D it encloses. Mathematically, it states:

$$
\oint_C (P\, dx + Q\, dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy
$$

where:

- P and Q are the components of the vector field $\mathbf{F} = P\, \mathbf{i} + Q\, \mathbf{j}$,
- C is the positively oriented, simple, closed curve bounding D.

The theorem effectively simplifies the evaluation of certain line integrals by converting them into double integrals, which may be easier to compute, especially when the region D has a convenient shape.

Why is the Left Side of Green’s Theorem Important?

The left side of Green's theorem—the line integral over C—represents the circulation of the vector field around the boundary. Evaluating this integral explicitly is often the first step in applying the theorem, especially when the double integral is complicated or when the goal is to understand the behavior of the field along the boundary.

The Given Vector Field: $\mathbf{F} = X\, \mathbf{A_x} + Y\, \mathbf{A_y}$

Understanding the Field Components

The vector field provided is:

$$
\mathbf{F} = X\, \mathbf{A_x} + Y\, \mathbf{A_y}
$$

which indicates:

- The component $P = X$,
- The component $Q = Y$.

Here, X and Y are the Cartesian coordinates, and $\mathbf{A_x}$, $\mathbf{A_y}$ are unit vectors in the x and y directions, respectively.

This field is linear in the coordinates:

$$
\mathbf{F}(X, Y) = (X, Y)
$$

which is a common example in vector calculus, representing a radial or position-based field.

Geometric and Physical Interpretation

The field $\mathbf{F} = (X, Y)$:

- Points radially outward from the origin,
- Is often associated with uniform expansion or divergence,
- Has a divergence $\nabla \cdot \mathbf{F} = \frac{\partial X}{\partial X} + \frac{\partial Y}{\partial Y} = 1 + 1 = 2$,
- Has curl $\nabla \times \mathbf{F} = 0$, since in two dimensions, the curl is:

$$
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} = 0 - 0 = 0
$$

Understanding these properties will be critical when applying Green's theorem.

Evaluating the Line Integral Along a Rectangular Path

Defining the Path C

Suppose the rectangular path C is specified by its vertices, often aligned with the axes for simplicity:

- (x_1, y_1),
- (x_2, y_1),
- (x_2, y_2),
- (x_1, y_2),

and the path proceeds counterclockwise.

For this analysis, assume a rectangle with vertices:

$$
(0, 0), \quad (a, 0), \quad (a, b), \quad (0, b)
$$

where $a, b > 0$.

The goal is to evaluate:

$$
\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\, dx + Q\, dy
$$

which involves integrating along each segment of the rectangle.

Step-by-Step Calculation

Let's examine each segment:

Segment 1: from $(0,0)$ to $(a, 0)$

- $y = 0$,
- x varies from 0 to a,
- $dy = 0$,
- dx varies from 0 to a.

Line integral:

$$
I_1 = \int_{x=0}^{a} P(x, 0) dx + \int_{x=0}^{a} Q(x, 0) \cdot 0 = \int_{0}^{a} x\, dx = \frac{a^2}{2}
$$

Segment 2: from $(a, 0)$ to (a, b)

- $x = a$,
- y varies from 0 to b,
- $dx = 0$,
- dy varies from 0 to b.

Line integral:

$$
I_2 = \int_{0}^{b} P(a, y) \cdot 0 + \int_{0}^{b} Q(a, y) dy = \int_{0}^{b} y\, dy = \frac{b^2}{2}
$$

Segment 3: from (a, b) to $(0, b)$

- $y = b$,
- x varies from a to 0,
- $dx = -dx'$,
- $dy = 0$.

Line integral:

$$
I_3 = \int_{x=a}^{0} P(x, b) dx + 0 = - \int_{0}^{a} x\, dx = - \frac{a^2}{2}
$$

Segment 4: from $(0, b)$ back to $(0, 0)$

- $x = 0$,
- y varies from b to 0,
- $dx = 0$,
- dy varies from b to 0.

Line integral:

$$
I_4 = 0 + \int_{b}^{0} Q(0, y) dy = - \int_{0}^{b} y\, dy = - \frac{b^2}{2}
$$

Summing the Contributions

Adding all four segments:

$$
\text{Total line integral} = I_1 + I_2 + I_3 + I_4 = \frac{a^2}{2} + \frac{b^2}{2} - \frac{a^2}{2} - \frac{b^2}{2} = 0
$$

This result indicates that the circulation of the field $\mathbf{F} = (X, Y)$ around this rectangle is zero.

Physical and Mathematical Implications

The zero value of the line integral aligns with the earlier calculation of the curl:

$$
\nabla \times \mathbf{F} = 0
$$

which suggests that the field is irrotational, and the circulation around any closed path should be zero in a simply connected domain, consistent with conservative vector fields.

Connecting to Green’s Theorem

Verifying the Theorem

Recall that Green’s theorem states:

$$
\oint_C (P\, dx + Q\, dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy
$$

Given $P = X$, $Q = Y$:

$$
\frac{\partial Q}{\partial x} = 0, \quad \frac{\partial P}{\partial y} = 0
$$

Thus,

$$
\oint_C (X\, dx + Y\, dy) = \iint_D 0\, dx\,dy = 0
$$

which matches the direct calculation, confirming consistency.

Significance of the Result

This demonstrates that for the vector field \(\

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