Given The Vector Valued Functions R(t) = Costi+ Sin Tj E^(2t)*kand U(t) = Ti+ Sin Tj + Costkcalculate

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Introduction

Vector calculus is a fundamental branch of mathematics that deals with vector-valued functions, which are functions that assign a vector to each point in a certain domain, typically time. These functions are essential in physics, engineering, and computer graphics, where they model phenomena such as motion, force fields, and more.

In this article, we will analyze and compute various properties of two specific vector-valued functions:
- $R(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + e^{2t}\, \mathbf{k}$
- $U(t) = t\, \mathbf{i} + \sin t\, \mathbf{j} + \cos t\, \mathbf{k}$

Our focus will be on understanding their derivatives, integrals, and geometric interpretations, with a detailed step-by-step approach suitable for students and enthusiasts seeking to deepen their understanding of vector calculus.

Understanding the Given Functions

Vector Function $R(t)$

The vector function:

$$
R(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + e^{2t}\, \mathbf{k}
$$

- Components:
- $x(t) = \cos t$
- $y(t) = \sin t$
- $z(t) = e^{2t}$

- Interpretation:
- The x and y components describe a circle of radius 1 in the xy-plane.
- The z-component grows exponentially, indicating that the curve spirals upward as t increases.

Vector Function $U(t)$

The vector function:

$$
U(t) = t\, \mathbf{i} + \sin t\, \mathbf{j} + \cos t\, \mathbf{k}
$$

- Components:
- $x(t) = t$
- $y(t) = \sin t$
- $z(t) = \cos t$

- Interpretation:
- The x-component increases linearly with t, representing motion along the x-axis.
- The y and z components trace a circle in the yz-plane, with radius 1, as t varies.

Derivatives of $R(t)$ and $U(t)$

Calculating derivatives helps us understand the velocity and acceleration vectors of the particle moving along these paths.

Derivative of $R(t)$

$$
R(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + e^{2t}\, \mathbf{k}
$$

Applying differentiation component-wise:

$$
R'(t) = -\sin t\, \mathbf{i} + \cos t\, \mathbf{j} + 2 e^{2t}\, \mathbf{k}
$$

Interpretation:
- The derivatives of the circular components are sinusoidal and cosine functions, indicating rotational motion.
- The exponential term's derivative introduces a multiplying factor of 2, affecting the z-direction velocity.

Derivative of $U(t)$

$$
U(t) = t\, \mathbf{i} + \sin t\, \mathbf{j} + \cos t\, \mathbf{k}
$$

Differentiating component-wise:

$$
U'(t) = \mathbf{i} + \cos t\, \mathbf{j} - \sin t\, \mathbf{k}
$$

\]

Interpretation:
- The velocity in the x-direction is constant (1), indicating uniform motion along x.
- The y and z components oscillate sinusoidally, representing circular motion in the yz-plane.

Calculating the Magnitudes of the Derivatives

The magnitude of the velocity vector $R'(t)$ and $U'(t)$ gives the speed of the particle along the curve.

Magnitude of $R'(t)$

$$
|R'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (2 e^{2t})^2}
$$
$$
|R'(t)| = \sqrt{\sin^2 t + \cos^2 t + 4 e^{4t}}
$$

Using the Pythagorean identity:

$$
\sin^2 t + \cos^2 t = 1
$$

Thus:

$$
|R'(t)| = \sqrt{1 + 4 e^{4t}}
$$

Magnitude of $U'(t)$

$$
|U'(t)| = \sqrt{(1)^2 + (\cos t)^2 + (\sin t)^2}
$$
$$
|U'(t)| = \sqrt{1 + \cos^2 t + \sin^2 t}
$$

Again, using the Pythagorean identity:

$$
|U'(t)| = \sqrt{1 + 1} = \sqrt{2}
$$

Summary:

- The speed along $R(t)$ varies with t, increasing exponentially.
- The speed along $U(t)$ is constant at $\sqrt{2}$.

Arc Length Calculations

The length of a curve between two points can be found by integrating the magnitude of the derivative over the interval.

Arc Length of $R(t)$ from $t = a$ to $t = b$

$$
L_R = \int_a^b |R'(t)|\, dt = \int_a^b \sqrt{1 + 4 e^{4t}}\, dt
$$

This integral generally requires substitution or numerical methods for evaluation, as it does not have an elementary antiderivative.

Arc Length of $U(t)$ from $t = a$ to $t = b$

$$
L_U = \int_a^b |U'(t)|\, dt = \int_a^b \sqrt{2}\, dt = \sqrt{2} (b - a)
$$

- Since the integrand is constant, the arc length is straightforward to compute.

Geometric Interpretations and Visualizations

Understanding the geometry of these curves enhances comprehension.

Geometry of $R(t)$

- The x and y components trace a circle of radius 1.
- The z component increases exponentially.
- Result: A spiral that wraps around the z-axis, ascending rapidly as t increases.

Geometry of $U(t)$

- The y and z components describe a circle of radius 1 in the yz-plane.
- The x component increases linearly.
- Result: A helix extending infinitely along the x-axis, with a circular cross-section in the yz-plane.

Calculating the Dot Product and Angle Between $R'(t)$ and $U'(t)$

Understanding the angle between the velocity vectors provides insight into the motion's directionality.

Dot Product

$$
R'(t) \cdot U'(t) = (-\sin t)(1) + (\cos t)(\cos t) + (2 e^{2t})(- \sin t)
$$
$$
R'(t) \cdot U'(t) = - \sin t + \cos^2 t - 2 e^{2t} \sin t
$$

Cosine of the angle $\theta(t)$

$$
\cos \theta(t) = \frac{R'(t) \cdot U'(t)}{|R'(t)| \, |U'(t)|}
$$

Using the previously computed magnitudes:

$$
|R'(t)| = \sqrt{1 + 4 e^{4t}}
$$
$$
|U'(t)| = \sqrt{2}
$$

The exact value of the angle depends on t, but this formula allows for numerical evaluation at specific points.

Applications and Significance in Physics and Engineering

The analysis of these vector functions is not purely theoretical; they model real-world phenomena:

- Particle motion: The functions represent paths of particles under various forces.
- Electromagnetic fields: The functions can model the trajectory of charged particles in magnetic fields.
- Robotics and computer graphics: Path planning and rendering involve calculating such curves and their properties.

Summary of Key Calculations

Property	$R(t)$	$U(t)$
Derivative $R'(t)$	$-\sin t\, \mathbf{i} + \cos t\, \mathbf{j} + 2 e^{2t}\, \mathbf{k}$	$\mathbf{i} + \cos t\, \mathbf{j} - \sin t\, \mathbf{k}$
Magnitude $\lvert R'(t)\rvert$	$\sqrt{1 + 4 e^{4t}}$	\(\	

Frequently Asked Questions

What is the primary goal when working with the vector functions $R(t) = \cos t\, i + \sin t\, j + e^{2t}\, k$ and $U(t) = t\, i + \sin t\, j + \cos t\, k$?

The main goal is to analyze properties such as their derivatives, tangent vectors, curvature, or to find relationships like dot or cross products between the two vector functions.

How do you compute the derivative of the vector function $R(t) = \cos t\, i + \sin t\, j + e^{2t}\, k$?

Differentiate each component separately: $R'(t) = -\sin t\, i + \cos t\, j + 2e^{2t}\, k$.

What is the significance of calculating the dot product $R(t) \cdot U(t)$ for these vector functions?

Calculating the dot product helps determine the angle between the vectors at a given t, or whether they are orthogonal, and provides insights into their spatial relationship.

How can we find the cross product $R(t) \times U(t)$ for the given functions, and what does it represent?

Use the determinant method to compute $R(t) \times U(t)$, which yields a vector perpendicular to both $R(t)$ and $U(t)$, indicating their orientation and potential for calculating area of the parallelogram they span.

What is the importance of analyzing the magnitude of these vector functions, such as $|R(t)|$ and $|U(t)|$?

The magnitudes provide information about the length or size of the vectors at each t, which is useful for understanding their behavior, normalization, or for calculating quantities like speed or arc length.

How can these vector functions be used to find the parametric equations of their tangent lines at a specific value of t?

Calculate the derivatives $R'(t)$ and $U'(t)$ to get the tangent vectors, then use the point at t to write the equations of the tangent lines in parametric form: point + t tangent vector.

Additional Resources

Given The Vector Valued Functions R(t) = Costi + Sin Tj + e^{2t}k and U(t) = Ti + Sin Tj + Cos Tk, the task of calculating various properties derived from these functions—such as their derivatives, magnitudes, dot products, cross products, and other related quantities—serves as a fundamental exercise in vector calculus with broad applications in physics and engineering. This article aims to explore these vector functions comprehensively, breaking down the calculations with clarity and technical precision while maintaining accessibility for readers with a basic understanding of calculus and vectors.

Introduction: Understanding the Vector Functions

The given vector functions are parametric representations of vector quantities depending on a parameter t:

- R(t): $R(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + e^{2t}\, \mathbf{k}$
- U(t): $U(t) = t\, \mathbf{i} + \sin t\, \mathbf{j} + \cos t\, \mathbf{k}$

These functions describe curves in three-dimensional space, with each component representing a coordinate that varies with t. The functions combine trigonometric, exponential, and linear terms, making their analysis rich and illustrative of key concepts in vector calculus.

Section 1: Derivatives of the Vector Functions

Derivatives of vector functions describe how these vectors change with respect to the parameter t. They are essential for understanding velocity, acceleration, tangent vectors, and the behavior of the curves.

1.1 Derivative of R(t)

Given:

$$
R(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + e^{2t}\, \mathbf{k}
$$

Differentiate each component:

- $\frac{d}{dt} (\cos t) = -\sin t$
- $\frac{d}{dt} (\sin t) = \cos t$
- $\frac{d}{dt} (e^{2t}) = 2e^{2t}$

Thus:

$$
R'(t) = -\sin t\, \mathbf{i} + \cos t\, \mathbf{j} + 2e^{2t}\, \mathbf{k}
$$

1.2 Derivative of U(t)

Given:

$$
U(t) = t\, \mathbf{i} + \sin t\, \mathbf{j} + \cos t\, \mathbf{k}
$$

Differentiate each component:

- $\frac{d}{dt} (t) = 1$
- $\frac{d}{dt} (\sin t) = \cos t$
- $\frac{d}{dt} (\cos t) = -\sin t$

Resulting in:

$$
U'(t) = \mathbf{i} + \cos t\, \mathbf{j} - \sin t\, \mathbf{k}
$$

Section 2: Magnitudes of the Vectors

Calculating the magnitude of a vector function at a point t provides insight into the length or size of the vector—crucial for understanding speed in physics or the scale of the vector in space.

2.1 Magnitude of R(t)

$$
|R(t)| = \sqrt{ (\cos t)^2 + (\sin t)^2 + (e^{2t})^2 }
$$

Using the Pythagorean identity:

$$
(\cos t)^2 + (\sin t)^2 = 1
$$

Hence:

$$
|R(t)| = \sqrt{1 + e^{4t}}
$$

This expression indicates that the magnitude of $R(t)$ depends heavily on the exponential term, which grows rapidly as t increases.

2.2 Magnitude of U(t)

$$
|U(t)| = \sqrt{ (t)^2 + (\sin t)^2 + (\cos t)^2 }
$$

Again, by Pythagoras:

$$
(\sin t)^2 + (\cos t)^2 = 1
$$

Thus:

$$
|U(t)| = \sqrt{ t^2 + 1 }
$$

This magnitude increases linearly with $|t|$, reflecting the unbounded linear component.

Section 3: Dot Product and Angle Between the Vectors

Understanding the angle between two vectors involves calculating their dot product and applying the cosine formula:

$$
\cos \theta = \frac{ R(t) \cdot U(t) }{ |R(t)|\, |U(t)| }
$$

3.1 Dot Product $R(t) \cdot U(t)$

Calculate:

$$
R(t) \cdot U(t) = (\cos t)(t) + (\sin t)(\sin t) + (e^{2t})(\cos t)
$$

Simplify:

$$
R(t) \cdot U(t) = t \cos t + \sin^2 t + e^{2t} \cos t
$$

3.2 Determining the Angle $\theta(t)$

The angle between $R(t)$ and $U(t)$:

$$
\theta(t) = \arccos \left(\frac{ t \cos t + \sin^2 t + e^{2t} \cos t }{ \sqrt{1 + e^{4t}} \times \sqrt{ t^2 + 1 } } \right)
$$

This expression provides the instantaneous angle between the two vectors at any parameter t.

Section 4: Cross Product of the Vectors

The cross product $R(t) \times U(t)$ yields a vector orthogonal to both $R(t)$ and $U(t)$, significant in understanding torque, angular momentum, and surface orientation.

4.1 Calculation of $R(t) \times U(t)$

Using the determinant form:

$$
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\cos t & \sin t & e^{2t} \\
t & \sin t & \cos t \\
\end{vmatrix}
$$

Compute each component:

- $\mathbf{i}$ component:

$$
(\sin t)(\cos t) - (e^{2t})(\sin t) = \sin t \cos t - e^{2t} \sin t = \sin t (\cos t - e^{2t})
$$

- $\mathbf{j}$ component:

$$
- [(\cos t)(\cos t) - (e^{2t})(t)] = - [\cos^2 t - t e^{2t}] = t e^{2t} - \cos^2 t
$$

- $\mathbf{k}$ component:

$$
(\cos t)(\sin t) - (\sin t)(t) = \cos t \sin t - t \sin t = \sin t (\cos t - t)
$$

Thus:

$$
R(t) \times U(t) = \sin t (\cos t - e^{2t})\, \mathbf{i} + (t e^{2t} - \cos^2 t)\, \mathbf{j} + \sin t (\cos t - t)\, \mathbf{k}
$$

Section 5: Curvature and Torsion (Optional Advanced Topics)

While the explicit calculation of curvature and torsion involves second derivatives and more advanced formulas, these quantities are critical in analyzing the geometric properties of the curves defined by $R(t)$ and $U(t)$.

5.1 Curvature $\kappa(t)$

Defined as:

$$
\kappa(t) = \frac{ | R'(t) \times R''(t) | }{ | R'(t) |^3 }
$$

5.2 Torsion $\tau(t)$

Calculated via:

$$
\tau(t) = \frac{ (R'(t) \times R''(t)) \cdot R'''(t) }{ | R'(t) \times R''(t) |^2 }
$$

These quantities provide insights into how sharply the curve bends and twists in space, respectively.

Conclusion: Synthesis and Applications

The detailed analysis of the vector functions $R(t)$ and $U(t)$ demonstrates the depth and versatility of vector calculus in understanding three-dimensional curves. From derivatives that describe velocities, to magnitudes indicating sizes, and dot/cross products revealing angles and orthogonality, each calculation contributes to a comprehensive understanding of the space these vectors inhabit.

Practically, such calculations underpin many physics and engineering disciplines. For example:

- In physics: Analyzing particle trajectories, electromagnetic fields, and flow dynamics.
- In engineering: Designing robotic arms, analyzing mechanical linkages, or modeling fluid flow.
- In computer graphics: Rendering curves and surfaces with realistic motion and shading.

Understanding and performing these calculations enhances our ability to model, analyze, and interpret complex spatial phenomena, bridging the gap between theoretical mathematics and real-world applications.

In summary, the analysis of \(R(t) \) and \(U(t) \) involves a

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