If An Exponential Function With Base 3 Has An Asymptote At Y=-6, And Passes Through The Point (2, 3),

If An Exponential Function With Base 3 Has An Asymptote At Y=-6, And Passes Through The Point (2, 3), understanding how to analyze and derive this function is essential for students and professionals working with exponential equations in mathematics, engineering, economics, and related fields. Exponential functions are fundamental in modeling growth, decay, and many real-world phenomena. Knowing how to determine their equations based on given conditions enables more accurate modeling and analysis.

In this comprehensive guide, we will explore the process of constructing the exponential function given its asymptote and a specific point it passes through. We'll delve into the mathematical background, step-by-step derivation, and practical applications. By the end of this article, you'll have a thorough understanding of how to approach similar problems and interpret exponential functions with vertical shifts and asymptotes.

Understanding Exponential Functions and Asymptotes

What Is an Exponential Function?

An exponential function is a mathematical expression of the form:

$$f(x) = a \cdot b^{x} + c$$

where:

- a is a coefficient that scales the function vertically.
- b is the base of the exponential, with $b > 0$ and $b \neq 1$.
- c is a vertical shift, moving the graph up or down.

The core exponential function, $y = b^{x}$, is increasing for $b > 1$ and decreasing for $0 < b < 1$. It is characterized by its rapid growth or decay and its smooth, continuous curve.

Role of Asymptotes in Exponential Functions

An asymptote is a line that the graph of a function approaches but never touches or crosses. For exponential functions, the asymptote is typically a horizontal line that the curve approaches as x approaches infinity or negative infinity.

In the general form:

$$f(x) = a \cdot b^{x} + c$$

the horizontal asymptote is:

$$y = c$$

This is because as $x \to \infty$ or $x \to -\infty$, the exponential term b^{x} either grows very large or approaches zero, depending on the base b. Specifically:

- If $b > 1$, then $b^{x} \to 0$ as $x \to -\infty$.
- If $0 < b < 1$, then $b^{x} \to 0$ as $x \to \infty$.

In either case, the asymptote is determined by the vertical shift c.

Problem Context and Objective

Given the problem:

- The exponential function has a base of 3 (i.e., $b = 3$).
- Its asymptote is at $y = -6$.
- It passes through the point $(2, 3)$.

Our goal is to find the explicit equation of this exponential function, understand its characteristics, and analyze its behavior.

Step-by-Step Derivation of the Function

Step 1: Write the General Form

Since the base is specified as 3, and the asymptote is at $y = -6$, the general form of the function becomes:

$$ f(x) = a \cdot 3^{x} + c $$

where:

- $c = -6$ (since the asymptote is at $y = -6$).

Thus,

$$ f(x) = a \cdot 3^{x} - 6 $$

Step 2: Use the Point to Find a

The function passes through $(2, 3)$, which means:

$$ f(2) = 3 $$

Substituting into the function:

$$ 3 = a \cdot 3^{2} - 6 $$

Calculate 3^{2}:

$$ 3^{2} = 9 $$

So:

$$ 3 = a \cdot 9 - 6 $$

Add 6 to both sides:

$$ 3 + 6 = a \cdot 9 $$

$$ 9 = 9a $$

Solve for a:

$$ a = \frac{9}{9} = 1 $$

Step 3: Write the Final Equation

With $a = 1$ and $c = -6$, the explicit form of the function is:

$$ \boxed{f(x) = 3^{x} - 6} $$

This is the exponential function satisfying all given conditions.

Analysis of the Derived Function

Graphical Behavior

- The function $f(x) = 3^{x} - 6$ is an exponential growth curve shifted downward by 6 units.
- The asymptote at $y = -6$ indicates that as $x \to \pm \infty$, the graph approaches this line without crossing it.
- For large positive x, 3^{x} becomes very large, so $f(x) \to \infty$.
- For large negative x, $3^{x} \to 0$, so $f(x) \to -6$.

Key Points and Intercepts

- Y-intercept: Find $f(0)$:

$$f(0) = 3^{0} - 6 = 1 - 6 = -5$$

The y-intercept is at $(0, -5)$.

- Point (2, 3): Confirmed by substitution.

- Behavior near asymptote: The graph approaches $y = -6$ as $x \to -\infty$.

Additional Insights and Applications

Understanding the Impact of the Vertical Shift

The vertical shift $c = -6$ demonstrates how adding or subtracting constants affects the asymptote position. In real-world scenarios:

- If modeling population growth, the asymptote might represent a lower bound or carrying capacity.
- In finance, it could symbolize a baseline level of income or expenditure.

Calculating Exponential Growth Rate

The base $b = 3$ indicates a rapid increase, with the quantity tripling for every unit increase in x. The growth rate r can be related to the base via:

$$b = e^{r} \Rightarrow r = \ln(b)$$

So,

$$r = \ln(3) \approx 1.0986$$

This indicates a growth rate of approximately 109.86% per unit increase in x.

Real-World Examples and Contexts

- Population Dynamics: Modeling populations that grow exponentially with a minimum baseline.
- Radioactive Decay or Decay Models: When the decay approaches a lower asymptote.
- Economics: Modeling investments or revenues that grow exponentially with shifts representing fixed costs or baseline revenues.

Common Mistakes and Clarifications

- Confusing the base and the asymptote: Remember, the asymptote is determined by the vertical shift c, not the base.
- Forgetting the negative shift: The function is shifted downward by 6 units, which affects the position of the asymptote.
- Miscalculating a: Always substitute the known point after setting the general form to find the coefficient.

Summary

- The exponential function with base 3, asymptote at $y = -6$, passing through $(2, 3)$, is:

$$\boxed{f(x) = 3^{x} - 6}$$

- The key steps involved recognizing the form, using the given point to solve for the coefficient, and understanding the role of the asymptote as the horizontal line $y = c$.

- The graph exhibits exponential growth behavior, approaching $y = -6$ as $x \to -\infty$, and increasing rapidly for larger x.

Conclusion

Analyzing exponential functions with given asymptotes and points is an essential skill in mathematics. It allows for precise modeling of various phenomena and provides insight into the behavior of exponential growth and decay. By understanding the structure of these functions and how to derive their equations, students and professionals can better interpret data, develop models, and solve complex problems involving exponential relationships.

Whether you are studying pure mathematics, applying these concepts in science or economics, or working on practical data modeling, mastering the process showcased here will bolster your analytical toolkit and enhance your problem-solving capabilities.

Frequently Asked Questions

What is the general form of an exponential function with base 3 that has a horizontal asymptote at y = -6?

The general form is $y = a 3^x + k$, where k is the asymptote. Since the asymptote is at $y = -6$, the function is $y = a 3^x - 6$.

How can we determine the value of 'a' in the exponential function y = a 3^x - 6 that passes through (2, 3)?

Substitute x=2 and y=3 into the equation: $3 = a 3^2 - 6$. Simplify to $3 = a 9 - 6$, then add 6 to both sides: $9 = a 9$, so $a = 1$.

What is the complete equation of the exponential function passing through (2, 3) with an asymptote at y = -6?

The complete function is $y = 1 3^x - 6$, which simplifies to $y = 3^x - 6$.

How does the asymptote at y = -6 affect the graph of the exponential function?

The asymptote at y = -6 means the graph approaches but never crosses y = -6 as x approaches negative infinity, shifting the basic exponential curve downward by 6 units.

What is the significance of the point (2, 3) in determining the specific exponential function?

The point (2, 3) provides a specific coordinate that allows us to solve for the constant 'a', ensuring the exponential function passes through this point and is uniquely defined.

Additional Resources

If An Exponential Function With Base 3 Has An Asymptote At Y = -6, And Passes Through The Point (2, 3)

Introduction

In the realm of mathematical functions, exponential models stand out for their distinctive growth and decay behaviors, as well as their applications across sciences, engineering, and economics. A specific case involving an exponential function with base 3, which exhibits a horizontal asymptote at $y = -6$ and passes through the point $(2, 3)$, provides a fascinating exploration into the structure and properties of exponential functions. This investigation aims to dissect the characteristics of such a function, elucidate its formulation, and analyze its behavior through a comprehensive, rigorous approach suitable for academic review or scholarly publication.

Understanding the Framework of Exponential Functions

Exponential functions generally take the form:

$$y = a \cdot b^{x} + c$$

where:
- a controls the vertical stretch or compression,
- b is the base, determining the rate of exponential growth ($b > 1$) or decay ($0 < b < 1$),
- c shifts the graph vertically, often serving as a horizontal asymptote if the function tends toward $y = c$ as $x \to \pm \infty$.

Given the problem's parameters, the function has:
- Base $b = 3$
- Asymptote at $y = -6$

- Passes through $(2, 3)$

The central task involves finding the specific form of this exponential function and analyzing its properties.

Deriving the Function: Incorporating the Asymptote

The Role of the Horizontal Asymptote

In exponential functions, a horizontal asymptote at $y = c$ typically indicates that the function approaches c as $x \rightarrow \pm \infty$. For the given case, the asymptote at $y = -6$ suggests the general form:

$$y = a \cdot 3^{x} + k$$

where $k = -6$.

This simplifies the model to:

$$y = a \cdot 3^{x} - 6$$

The task now reduces to determining the coefficient a.

Determining the Coefficient a

Given the point $(2, 3)$ lies on the graph, substitute $x = 2$ and $y = 3$:

$$
3 = a \cdot 3^{2} - 6
$$

Calculating 3^{2}:

$$
3^{2} = 9
$$

Thus,

$$
3 = a \cdot 9 - 6
$$
$$
3 + 6 = 9a
$$
$$
9 = 9a

\]

$$
a = 1
$$

Result:

$$
\boxed{
y = 3^{x} - 6
}
$$

This is the explicit form of the exponential function satisfying the given conditions.

Verifying the Function's Properties

1. Asymptotic Behavior

As $x \to \infty$:

$$
3^{x} \to \infty
$$

$$
y \to \infty
$$

As $x \to -\infty$:

$$
3^{x} \to 0
$$

$$
y \to -6
$$

which confirms that the horizontal asymptote at $y = -6$ is valid and the function approaches this line from above as $x \to -\infty$.

2. Growth Characteristics

Since $b = 3 > 1$, the function exhibits exponential growth for increasing x. The coefficient $a = 1$ indicates a standard amplitude, with no additional scaling, and the vertical shift ensures the asymptote is at $y = -6$.

3. Point Verification

Plugging $x = 2$:

$$
y = 3^{2} - 6 = 9 - 6 = 3
$$

which confirms the point $(2, 3)$ lies on the graph.

Deep Dive: Graphical and Analytical Characteristics

Graphical Behavior

The graph of $y = 3^{x} - 6$:

- Passes through $(2, 3)$,
- Approaches $y = -6$ from above as $x \to -\infty$,
- Rises rapidly for large x,
- Is strictly increasing across its domain $(-\infty, \infty)$,
- Has no x-intercepts (since 3^{x} is always positive, y is always greater than -6).

Critical Points and Derivatives

- Derivative:

$$
\frac{dy}{dx} = \frac{d}{dx} (3^{x} - 6) = 3^{x} \ln 3
$$

which is always positive, indicating the function is monotonically increasing.

- Inflection points:

Since the second derivative:

$$
\frac{d^{2} y}{dx^{2}} = 3^{x} (\ln 3)^2 > 0
$$

for all x, the function is convex everywhere.

Broader Implications and Applications

Understanding such exponential functions is crucial in modeling phenomena where growth occurs above a baseline or with a vertical shift, such as:

- Population dynamics with a minimum baseline,
- Radioactive decay with an offset,
- Investment growth with a fixed baseline return.

In particular, the explicit identification of the asymptote and passing point facilitates strategic modeling in fields requiring precise calibration of exponential functions.

Summary of Key Findings

- The exponential function with base 3, asymptote at \(y = -6 \), passing through \((2, 3) \), is:

\[
\boxed{
y = 3^{x} - 6
}
\]

- It has a horizontal asymptote at \(y = -6 \),
- It passes through the point \((2, 3) \),
- It exhibits strictly increasing behavior,
- The derivative confirms continuous growth without inflection points.

Conclusion

The investigative analysis into the exponential function \(y = 3^{x} - 6 \) reveals a straightforward yet instructive example of how asymptotes and specific points determine the precise form of exponential models. Understanding this function's properties not only deepens comprehension of exponential behavior but also underscores the importance of asymptotic analysis in mathematical modeling. Such detailed examination provides a foundation for advanced studies in mathematical functions and their real-world applications, emphasizing the elegance and utility of exponential functions in scientific inquiry.

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