If In = 5 X"" (1+x)0.5 Dx ,n= 0,1,2.... Where The Integral Is For X E [0,1]. By First Finding A Bound

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Understanding and evaluating integrals involving variable powers and parameters is fundamental in mathematical analysis, especially in the context of sequences and series. The integral expression:

$$
I_n = 5 \int_0^1 x^n (1 + x)^{0.5} \, dx
$$

where $n=0,1,2,\dots$, presents an interesting case for analysis. To analyze such an integral comprehensively, the initial step involves establishing bounds for I_n. Finding bounds provides valuable insight into the behavior of the sequence $\{ I_n \}$, especially as n tends to infinity. Bounding integrals is a crucial technique in mathematical analysis, serving as a foundational step before delving into explicit evaluations or asymptotic analyses.

This article aims to explore the integral in detail, starting with the process of establishing bounds for I_n, followed by methods to evaluate or approximate it, and finally discussing the implications of these bounds in the broader context of integral sequences.

Understanding the Integral I_n and Its Domain

Before deriving bounds, it is essential to understand the integral's structure and properties:

- Definition:

$$
I_n = 5 \int_0^1 x^n (1 + x)^{0.5} \, dx
$$

- Domain of integration: $x \in [0, 1]$

- Parameter n: Non-negative integers $n=0,1,2,\dots$

- Integrand properties:

- For $x \in [0,1]$, x^n is non-negative and decreasing as n increases.
- $(1 + x)^{0.5}$ is increasing on $[0, 1]$, with minimum at $x=0$ and maximum at $x=1$.

Understanding these properties allows us to establish meaningful bounds for I_n.

Establishing Bounds for I_n

Bounding the integral I_n involves finding functions $L(x)$ and $U(x)$ such that:

$$
L(x) \leq (1 + x)^{0.5} \leq U(x), \quad \forall x \in [0, 1]
$$

which then translate into bounds for I_n:

$$
5 \int_0^1 x^n L(x) \, dx \leq I_n \leq 5 \int_0^1 x^n U(x) \, dx
$$

The key steps are:

1. Identify simple bounds for $(1 + x)^{0.5}$ on $[0, 1]$.

2. Use these bounds to derive inequalities for I_n.

Bounding $(1 + x)^{0.5}$: Basic Inequalities

Since $(1 + x)^{0.5}$ is increasing on $[0,1]$, the minimum and maximum are straightforward:

- Lower bound:

$$
(1 + 0)^{0.5} = 1^{0.5} = 1
$$

- Upper bound:

$$
(1 + 1)^{0.5} = 2^{0.5} = \sqrt{2} \approx 1.4142
$$

Thus,

$$
1 \leq (1 + x)^{0.5} \leq \sqrt{2}, \quad \forall x \in [0,1]
$$

This gives initial bounds for the integral:

$$
5 \int_0^1 x^n \cdot 1 \, dx \leq I_n \leq 5 \int_0^1 x^n \sqrt{2} \, dx
$$

which simplifies to:

$$
5 \int_0^1 x^n \, dx \leq I_n \leq 5 \sqrt{2} \int_0^1 x^n \, dx
$$

Calculating the Basic Bounds

The integral of x^n over $[0,1]$ is well-known:

$$
\int_0^1 x^n \, dx = \frac{1}{n+1}
$$

Therefore, the bounds become:

$$
5 \cdot \frac{1}{n+1} \leq I_n \leq 5 \sqrt{2} \cdot \frac{1}{n+1}
$$

or explicitly,

$$
\boxed{
\frac{5}{n+1} \leq I_n \leq \frac{5 \sqrt{2}}{n+1}
}
$$

Implication: For all $n \geq 0$, the integral I_n is bounded between these two simple expressions. The bounds decay as $1/(n+1)$, giving a clear understanding of the asymptotic behavior as $n \to \infty$.

Refined Bounds Using Tighter Inequalities

While the basic bounds are useful, they are somewhat crude because they rely on the extremal values of $(1 + x)^{0.5}$ at the endpoints. To obtain tighter bounds, we can consider the mean value theorem or inequalities that better approximate $(1 + x)^{0.5}$.

Using the Average Value of $(1 + x)^{0.5}$ on $[0,1]$

The average value of the integrand $(1 + x)^{0.5}$ over $[0,1]$ is:

$$
\frac{1}{1-0} \int_0^1 (1 + x)^{0.5} \, dx
$$

Calculating this integral gives:

$$
\int_0^1 (1 + x)^{0.5} \, dx
$$

which can be computed explicitly:

$$
\int (1 + x)^{0.5} \, dx = \frac{2}{3} (1 + x)^{1.5} + C
$$

Hence,

$$
\int_0^1 (1 + x)^{0.5} \, dx = \frac{2}{3} \left[(1 + 1)^{1.5} - (1 + 0)^{1.5} \right]
$$

$$
= \frac{2}{3} \left[2^{1.5} - 1^{1.5} \right] = \frac{2}{3} \left[2^{1.5} - 1 \right]
$$

Recall that:

$$
2^{1.5} = 2^{1} \times 2^{0.5} = 2 \times \sqrt{2} \approx 2 \times 1.4142 = 2.8284
$$

Therefore,

$$
\int_0^1 (1 + x)^{0.5} \, dx \approx \frac{2}{3} (2.8284 - 1) = \frac{2}{3} \times 1.8284 \approx 1.2189
$$

The average value:

$$
\bar{f} = \frac{1.2189}{1} = 1.2189
$$

\]

Using this, we can refine bounds:

$$
I_n \approx 5 \times \bar{f} \times \int_0^1 x^n \, dx = 5 \times 1.2189 \times \frac{1}{n+1}
$$

which yields:

$$
I_n \approx \frac{6.0945}{n+1}
$$

This suggests that the actual I_n is close to this value, and the bounds can be tightened as:

$$
\frac{5}{n+1} \leq I_n \leq \frac{6.0945}{n+1}
$$

or, more generally,

$$
\boxed{
\frac{5}{n+1} \leq I_n \leq \left(5 \times \frac{2}{3} (2^{1.5} - 1) \right) \times \frac{1}{n+1}
}
$$

which provides a better approximation.

Asymptotic Behavior of I_n

Having established bounds, understanding the asymptotic behavior of I_n as $n \to \infty$ is crucial:

- Since I_n is bounded between $\frac{5}{n+1}$ and approximately $\frac{6.0945}{n+1}$, it follows that:

\[
I_n \sim

Frequently Asked Questions

What is the integral expression given in the problem, and what are its limits?

The integral is $In = 5 \times \int_0^1 (1 + x)^{0.5} dx$ for n = 0,1,2,..., with the limits of integration from 0 to 1.

Why is finding a bound important for evaluating the integral In?

Finding a bound helps estimate the maximum and minimum possible values of the integral, which is useful for understanding the behavior of In and for approximations or convergence analysis.

How can we determine an upper bound for the integral In?

Since $(1 + x)^{0.5}$ is increasing on [0,1], its maximum at x=1 is $\sqrt{2}$, so the integral is bounded above by $5 \times (\sqrt{2}) \times 1 = 5\sqrt{2}$.

What is the lower bound of the integral In over the interval [0,1]?

The minimum value of $(1 + x)^{0.5}$ on [0,1] is at x=0, which is 1, so the integral is bounded below by $5 \times 1 \times 1 = 5$.

How does the bound help in analyzing the behavior of In as n varies?

The bounds provide fixed limits within which In must lie, allowing us to analyze whether In converges, diverges, or remains bounded as n increases, especially when considering related series or asymptotic behavior.

Can the bounds be used to approximate the value of In?

Yes, the bounds give a range for In, and with more detailed analysis or numerical methods, they can guide approximations of the integral's actual value.

Additional Resources

Exploring the Integral: A Deep Dive into $\int_{0}^{1} 5x^{n} \sqrt{1 + x} \, dx$ and Its Bounds

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Introduction: The Significance of the Integral in Mathematical Analysis

Mathematics often presents us with intricate integrals that challenge our understanding and computational skills. Among these, the integral

$$
I(n) = \int_{0}^{1} 5x^{n} \sqrt{1 + x} \, dx
$$

where $n = 0, 1, 2, \dots$, stands out for its blend of polynomial and radical functions. This integral not only exemplifies the richness of calculus but also finds applications across fields such as physics, engineering, and probability theory, where understanding the behavior of such functions over the interval $[0, 1]$ is crucial.

In this article, we undertake a comprehensive exploration of $I(n)$, starting with establishing bounds for the integral, then delving into its analytical evaluation, and finally discussing the implications of its behavior as n varies. Our approach emphasizes understanding the integral's nature by first bounding it, a foundational step that informs further analysis and computational approximation.

Understanding the Structure of the Integral

Before diving into bounds and calculations, it’s vital to dissect the components of the integral:

$$
I(n) = \int_{0}^{1} 5x^{n} \sqrt{1 + x} \, dx
$$

Key elements:

- The polynomial term x^n: For $n \geq 0$, as n increases, x^n tends to be more concentrated near zero, affecting the integral's value.
- The radical $\sqrt{1 + x}$: This is a smooth, increasing function on $[0, 1]$, ranging from $\sqrt{1}$ at $x=0$ to $\sqrt{2}$ at $x=1$.
- The constant multiplier 5: Serves as a scaling factor, simplifying the bounds and analysis.

The integral’s domain, $[0,1]$, is bounded, and the integrand is continuous and positive over this interval, allowing us to employ various bounding techniques confidently.

Establishing a Bound: The First Step

Why Bound the Integral?

Bounding the integral is a fundamental step in approximation and analysis. It provides an estimate of the integral’s magnitude without requiring explicit evaluation, especially useful when dealing with complex integrands or when seeking asymptotic behavior.

Step 1: Identify the Range of the Integrand

Since $x \in [0, 1]$:

- $x^n \in [0, 1]$, with x^n decreasing as n increases for fixed $x \in (0,1)$.
- $\sqrt{1 + x} \in [1, \sqrt{2}]$.

Therefore, the integrand:

$$
f(x) = 5 x^{n} \sqrt{1 + x}
$$

satisfies:

$$
0 \leq f(x) \leq 5 \times 1 \times \sqrt{2} = 5 \sqrt{2} \approx 7.071
$$

for all $x \in [0,1]$. But to find meaningful bounds, especially in relation to n, we need to consider the behavior of x^n within the interval.

Step 2: Bounding x^n over $[0,1]$

For $x \in [0,1]$, the following inequalities hold:

$$
0 \leq x^{n} \leq 1
$$

- When $n=0$, $x^0=1$.
- For $n \geq 1$, $x^n \leq x^{n-1}$, but more precisely:

$$
x^n \leq x^{n-1} \quad \text{for } x \in (0,1)
$$

which indicates that x^n diminishes as n increases, especially near 0.

Given this, the integrand is bounded:

$$
0 \leq f(x) \leq 5 \sqrt{1 + x}

\]

with the maximum attained at \(x=1\):

\[
f(1) = 5 \times 1^n \times \sqrt{2} = 5 \sqrt{2} \approx 7.071
\]

and the minimum at \(x=0\):

\[
f(0) = 0
\]

Step 3: Deriving Explicit Bounds for \(I(n)\)

Using the bounds of the integrand, we can establish bounds for the integral:

\[
\boxed{
0 \leq I(n) \leq \int_{0}^{1} 5 \sqrt{1 + x} \, dx
}
\]

Calculating the upper bound:

\[
I(n) \leq 5 \int_{0}^{1} \sqrt{1 + x} \, dx
\]

This integral is straightforward:

\[
\int \sqrt{1 + x} \, dx
\]

Let's compute this indefinite integral:

\[
\int \sqrt{1 + x} \, dx
\]

Set:

\[
u = 1 + x \Rightarrow du = dx
\]

then:

$$
\int \sqrt{u} \, du = \frac{2}{3} u^{3/2} + C
$$

Thus:

$$
\int_{0}^{1} \sqrt{1 + x} \, dx = \left. \frac{2}{3} (1 + x)^{3/2} \right|_{0}^{1} = \frac{2}{3} \left[(2)^{3/2} - (1)^{3/2}\right]
$$

Calculate:

$$
(2)^{3/2} = 2^{1} \times 2^{1/2} = 2 \times \sqrt{2} \approx 2 \times 1.4142 = 2.8284
$$

Therefore:

$$
\int_{0}^{1} \sqrt{1 + x} \, dx = \frac{2}{3} (2.8284 - 1) = \frac{2}{3} \times 1.8284 \approx 1.2189
$$

Finally, the bound:

$$
I(n) \leq 5 \times 1.2189 \approx 6.0945
$$

Summary of bounds:

$$
0 \leq I(n) \leq \boxed{6.0945}
$$

for all $n \geq 0$.

Refining the Bounds: Considering the Behavior of x^{n}

While the previous bounds are valid, they are quite broad. To refine them, consider the influence of x^n for various n.

Case 1: $n=0$

Here:

$$
I(0) = \int_{0}^{1} 5 \sqrt{1 + x} \, dx \approx 6.0945
$$

which matches our upper bound, as $x^0=1$ simplifies the integral.

Case 2: $n \to \infty$

As n increases, x^n tends to zero for all $x \in [0,1)$, and equals 1 at $x=1$. For large n:

$$
I(n) \approx 5 \times 1^{n} \times \sqrt{2} \times \text{(small contribution from }x<1\text{)} \to 0
$$

since the dominant contribution shrinks toward zero as x^n decays rapidly inside $[0,1)$.

Case 3: $n=1$

The integral simplifies to:

$$
I(1) = 5 \int_{0}^{1} x \sqrt{1 + x} \, dx
$$

which can be evaluated explicitly, as shown later.

Analytical Evaluation of $I(n)$

Having established bounds, we now turn to explicit evaluation or approximation methods, focusing on techniques like substitution and Beta functions.

Method 1: Substitution Approach

Consider the integral:

$$
I(n) = 5 \int_{0}^{1} x^{n} \sqrt{1 + x} \, dx
$$

Let’s attempt a substitution to simplify the radical:

$$
u = 1 + x \Rightarrow x = u - 1, \quad du = dx
$$

When $x=0$, $u=1$, and when \(

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