If The Mean Of 5 Positive Integers Is 15, What Is The Maximum Possible Difference Between The Largest

If The Mean Of 5 Positive Integers Is 15, What Is The Maximum Possible Difference Between The Largest is a problem that combines the fundamentals of arithmetic, number theory, and strategic reasoning to explore how constraints influence the scope of possible values within a set. At first glance, it appears straightforward: we have five positive integers, their average is 15, and we want to find the maximum difference between the largest and smallest numbers in the set. However, the question invites a deeper exploration into how the sum, individual values, and their arrangement impact the maximum possible difference. Understanding these relationships not only enhances problem-solving skills but also deepens comprehension of how constraints shape possibilities in mathematical scenarios.

Understanding the Problem: Breaking Down the Key Elements

Before diving into the solution, it’s important to understand what the problem is asking and the constraints involved.

The Given Data

- Number of integers: 5
- Type of integers: Positive integers (meaning each is ≥ 1)
- Average (mean): 15

From the average, we can determine the total sum of the five integers:
- Total sum = average × number of integers = 15 × 5 = 75

Thus, the set of five positive integers must sum to 75.

The Goal

Maximize the difference between the largest and smallest integers in the set.

Mathematically, if the integers are a_1, a_2, a_3, a_4, a_5, with $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$, then:

\[

$$
\text{Difference} = a_5 - a_1
$$

Our task is to choose these integers to maximize $a_5 - a_1$, given:
- $a_i \in \mathbb{Z}^+$ (positive integers)
- $\sum_{i=1}^5 a_i = 75$

Approach to the Problem: Strategies for Maximization

Maximizing the difference between the largest and smallest integers involves a strategic allocation of the total sum. The key idea is to:
- Make the largest number as large as possible.
- Make the smallest number as small as possible (which is 1, since they are positive integers).

However, because the total sum is fixed at 75, increasing the largest number reduces the sum available for the other numbers, and vice versa.

Key Observations
- To maximize $a_5 - a_1$, we should set a_1 as small as possible, which is 1.
- Then, the sum allocated to the remaining four integers is $75 - a_1 = 74$.
- To maximize a_5, we should allocate as much of the remaining sum as possible to a_5, but since the other integers must be positive, the remaining four integers should be as small as possible, ideally all equal to 1.

Step-by-Step Solution

Let's formalize the process to find the maximum difference.

Step 1: Set the Smallest Integer
- Since the integers are positive, the smallest possible value for a_1 is 1.

Step 2: Minimize the Other Four Integers

- To allow a_5 to be as large as possible, set the other four integers to the minimum, which is 1 each:

$$
a_2 = a_3 = a_4 = 1
$$

- Sum of these four integers:

$$
a_2 + a_3 + a_4 + a_1 = 1 + 1 + 1 + 1 = 4
$$

- Remaining sum for a_5:

$$
a_5 = 75 - 4 = 71
$$

Step 3: Check Validity

- All integers are positive:
- $a_1 = 1$
- $a_2 = a_3 = a_4 = 1$
- $a_5 = 71$
- Sum check:

$$
1 + 1 + 1 + 1 + 71 = 75
$$

- The set is valid.

Step 4: Calculate the Difference

- Largest minus smallest:

$$
a_5 - a_1 = 71 - 1 = 70
$$

This suggests the maximum possible difference is 70.

Verifying the Maximality of the Result

Is it possible to achieve a larger difference? Let's consider the implications.

Attempting to Increase a_5

- The total sum is fixed at 75.
- The minimal sum for the other four integers is 4 (if each is 1).
- Increasing any of the other four integers above 1 would decrease the sum available for a_5, thus reducing its maximum potential.

Attempting to Decrease a_1

- Since a_1 must be a positive integer, the smallest possible is 1.
- It cannot be smaller.

Conclusion

- The configuration with $a_1 = 1$ and the other four integers at 1, with $a_5 = 71$, yields the maximum difference of 70.
- Any deviation from setting $a_1 = 1$ or the other four integers at their minimum would decrease the maximum a_5, thus decreasing the difference.

Additional Considerations: Alternative Configurations

While the above approach seems optimal, let's briefly consider other configurations to reinforce the conclusion.

Case 1: Slightly Larger Smallest Integer

- Suppose $a_1 = 2$.
- Then, the other four integers minimally could be 2 each:

$$
a_2 = a_3 = a_4 = 2
$$

- Sum of the four:

\[

$$
2 + 2 + 2 + 2 = 8
$$

- Remaining sum for a_5:

$$
75 - 8 = 67
$$

- Difference:

$$
a_5 - a_1 = 67 - 2 = 65
$$

- Which is less than 70.

Similarly, increasing a_1 further reduces the maximum a_5, decreasing the difference.

Conclusion: The maximum difference occurs when a_1 is minimized at 1.

Final Answer and Summary

Based on the above analysis, the maximum possible difference between the largest and smallest integers in a set of five positive integers averaging 15 (and summing to 75) is achieved when:
- The smallest integer is 1.
- The remaining four integers are also as small as possible (each 1), to maximize the remaining sum for the largest integer.
- The largest integer then takes the value $75 - (1+1+1+1) = 71$.

Therefore, the maximum possible difference is:

$$
\boxed{70}
$$

Key Takeaways for Problem Solving and Mathematical Reasoning

- Always leverage the constraints to set boundary values—in this case, setting the smallest integer to its minimum (1).
- Maximize the variable of interest (the largest integer) by minimizing the others, respecting the positivity constraint.
- Verify the sum constraints after choosing boundary values to ensure validity.
- Consider alternative configurations to confirm the optimal solution, but recognize that boundary values often lead to extremal solutions in such problems.

Understanding how to approach such problems enhances critical thinking and mathematical intuition, especially in optimization scenarios under constraints. The key is balancing the variables to push the boundaries of what is feasible, always respecting the fundamental rules of the problem.

Frequently Asked Questions

What is the total sum of the five positive integers if their mean is 15?

The total sum is 5 times 15, which equals 75.

How can we maximize the difference between the largest and smallest integers in this set?

To maximize the difference, we should minimize the smallest integers and maximize the largest integer while keeping the total sum at 75.

What is the smallest possible value for the four smallest integers to maximize the difference?

Since they are positive integers, the smallest value each can have is 1.

What would be the value of the largest integer if the four smallest are all 1?

The largest integer would be 75 minus the sum of the four smallest (which is 4), so 75 - 4 = 71.

What is the maximum possible difference between the largest and smallest integers in this scenario?

The maximum difference is 71 - 1 = 70.

Can the smallest integers be zero to maximize the difference?

No, since the integers are positive, the smallest they can be is 1.

Is it necessary for all four smallest integers to be equal to achieve the maximum difference?

No, but setting all four to 1 minimizes their sum and maximizes the largest integer, which achieves the maximum difference.

Could the largest integer be greater than 71 in this scenario?

No, because the total sum is fixed at 75, and with four integers at minimum 1, the largest cannot exceed 71.

What is the reasoning behind choosing the four smallest integers as 1?

Choosing the smallest positive integers (1) minimizes their total sum, allowing the largest integer to be as large as possible within the sum constraint, thus maximizing the difference.

What is the final answer to the maximum possible difference between the largest integers under the given conditions?

The maximum possible difference is 70.

Additional Resources

If the mean of 5 positive integers is 15, what is the maximum possible difference between the largest? This question is an intriguing blend of basic arithmetic, strategic reasoning, and problem-solving skills. It invites us to analyze a set of five positive integers, understand the implications of their average, and manipulate their values to maximize the difference between the largest and smallest integers in the set. Such problems are common in math competitions, standardized tests, and logical reasoning exercises, making them valuable for developing critical thinking and number sense.

In this comprehensive guide, we will explore how to approach this problem systematically, break down the reasoning process, and demonstrate how to arrive at the maximum possible difference between the largest and smallest integers under the given conditions.

Understanding the Problem

Let's carefully interpret the problem statement:

- We have five positive integers: call them a_1, a_2, a_3, a_4, a_5.
- The mean (average) of these integers is 15.
- The goal is to determine the maximum possible difference between the largest and smallest integers in this set.

Key points:

- Since the integers are positive, each $a_i \geq 1$.
- The mean being 15 implies:

$$
\frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} = 15
$$

Therefore:

$$
a_1 + a_2 + a_3 + a_4 + a_5 = 75
$$

- To maximize the difference between the largest and smallest integers, we need to understand how to assign values to the five integers to achieve this.

Basic Constraints and Observations

Before diving into strategies, let's clarify what constraints we face:

1. Positivity:
- All five integers are greater than or equal to 1.
2. Sum fixed:
- The total sum of the five integers must be exactly 75.
3. Maximizing difference:
- The difference between the largest and smallest integers is:

$$
\text{Difference} = a_{max} - a_{min}
$$

- To maximize this difference, intuitively, we want the smallest possible a_{min} (which is at least 1) and the largest possible a_{max}.

Strategy for Maximizing the Difference

Given the constraints, here's how to approach the problem:

- Set the smallest integer as low as possible to allow the largest integer to be as large as possible. Since the integers are positive, the smallest possible value is 1.
- Allocate the sum accordingly:
- Assign $a_{min} = 1$.
- Let a_{max} be as large as possible.
- The remaining three integers should be chosen to satisfy the sum and positivity constraints.

Step-by-Step Solution Outline

Let's formalize the reasoning:

Step 1: Fix the smallest integer

Set:

$$
a_{min} = 1
$$

Step 2: Express the sum of the remaining four integers

Since the total sum is 75, the sum of the remaining four integers is:

$$
a_2 + a_3 + a_4 + a_5 = 75 - a_{min} = 75 - 1 = 74
$$

- Among these four integers, one is the largest, a_{max}.
- The other three integers are positive and less than or equal to a_{max}.

Step 3: Maximize a_{max}

To maximize a_{max}, consider the following:

- The sum of the other four integers must be 74.
- To allow for the largest possible a_{max}, assign the other three integers as small as possible (since they are positive integers, the minimum

is 1).

Thus, the minimal sum for the other three integers is:

$$
\text{Sum of three smallest positive integers} = 1 + 1 + 1 = 3
$$

- The remaining sum for the largest integer is:

$$
a_{max} = 74 - 3 = 71
$$

- Now, verify if this is valid:

- The set of integers could be:

$$
a_1 = 1, \quad a_2 = 1, \quad a_3 = 1, \quad a_4 = 1, \quad a_5 = 71
$$

- The sum:

$$
1 + 1 + 1 + 1 + 71 = 75
$$

- The mean:

$$
\frac{75}{5} = 15
$$

- All are positive integers, satisfying the problem's conditions.

- The difference:

$$
71 - 1 = 70
$$

Final Result and Interpretation

By following this reasoning, the maximum possible difference between the largest and smallest integers is 70.

Answer: 70

Additional Considerations

While the above demonstrates a straightforward approach, let's explore some variations and potential pitfalls.

Could the smallest integer be greater than 1?

- If we increase the smallest integer a_{min}, the maximum a_{max} decreases because the total sum of the remaining integers decreases.
- For example, if $a_{min} = 2$, then the sum of the remaining four integers is:

$$
75 - 2 = 73
$$

- Assigning three of these integers as small as possible (all 1s):

$$
1 + 1 + 1 = 3
$$

- The largest integer:

$$
a_{max} = 73 - 3 = 70
$$

- The difference:

$$
70 - 2 = 68
$$

- Which is less than 70, confirming that choosing $a_{min} = 1$ yields the maximum difference.

Could the integers be non-distinct?

- Yes, the problem states positive integers, and they can be equal.
- Our approach remains valid regardless, as we're trying to maximize the difference, which is achieved when the integers are as spread out as possible.

Summary of Key Points

- To maximize the difference between the largest and smallest integers given

the mean, set the smallest integer to the minimum value (1).
- Allocate the remaining sum to maximize the largest integer while keeping the other integers as small as possible (at 1).
- The maximum largest integer in this scenario is 71.
- The maximum difference is:

$$
\boxed{71 - 1 = 70}
$$

Final Thoughts

This problem underscores the importance of strategic distribution of values within a fixed sum to achieve extremal differences. It combines fundamental concepts from arithmetic, inequalities, and logical reasoning.

In conclusion:

> If the mean of 5 positive integers is 15, the maximum possible difference between the largest and smallest integers is 70.

This problem exemplifies how understanding constraints and employing strategic assumptions can lead to elegant solutions in mathematical reasoning. Whether it's in competitions, exams, or real-world applications, these problem-solving techniques are invaluable tools in your mathematical toolkit.

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