Integrate The Function F = X 3y+ Z Over The Line Segment From The Point (0,0,0) To The Point (1,1,1).

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When dealing with line integrals in multivariable calculus, understanding how to evaluate integrals of scalar functions along a specified curve is essential. One common problem involves integrating a function over a line segment connecting two points in three-dimensional space. In this article, we will explore how to evaluate the line integral of the scalar function $F(x, y, z) = x^3 y + z$ along the line segment from the point $(0, 0, 0)$ to $(1, 1, 1)$. This step-by-step approach will deepen your understanding of parameterization, setting up the integral, and performing the calculations necessary to arrive at the solution.

Understanding the Problem

The goal is to compute the line integral:

$$
\int_{C} F(x, y, z) \, ds
$$

where C is the line segment from $(0, 0, 0)$ to $(1, 1, 1)$, and the scalar function is:

$$
F(x, y, z) = x^3 y + z
$$

This integral essentially sums the values of F along the path C, weighted by the differential arc length ds.

Key Concepts in Line Integrals

Before delving into the calculation, let's review some fundamental concepts:

1. Parameterization of a Curve

To evaluate a line integral, we first parameterize the curve C. For a line segment between two points, a simple parameterization uses a linear interpolation:

$$
\mathbf{r}(t) = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0), \quad t \in [0, 1]
$$

where:

- $\mathbf{r}_0 = (0, 0, 0)$
- $\mathbf{r}_1 = (1, 1, 1)$

Thus,

$$
\mathbf{r}(t) = (0 + t(1 - 0), 0 + t(1 - 0), 0 + t(1 - 0)) = (t, t, t)
$$

This parameterization is valid for $t \in [0, 1]$.

2. Computing ds

The differential arc length ds can be expressed in terms of the parameter t:

$$
ds = |\mathbf{r}'(t)| \, dt
$$

where

$$
\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}
$$

and

$$
|\mathbf{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}
$$

Step-by-Step Solution

Let's now proceed step-by-step to evaluate the integral.

Step 1: Parameterize the Curve

As established:

$$
\mathbf{r}(t) = (t, t, t), \quad t \in [0, 1]
$$

which yields:

\[
x = t, \quad y = t, \quad z = t
\]

Step 2: Find \(\mathbf{r}'(t) \) and \(|\mathbf{r}'(t)| \)

Differentiating each component:

\[
\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = 1
\]

Therefore,

\[
|\mathbf{r}'(t)| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}
\]

Step 3: Rewrite the Scalar Function \(F \) in Terms of \(t \)

Substitute the parameterization into \(F \):

\[
F(\mathbf{r}(t)) = (t)^3 \times t + t = t^4 + t
\]

Step 4: Set Up the Line Integral

The line integral becomes:

\[
\int_{t=0}^{1} F(\mathbf{r}(t)) |\mathbf{r}'(t)| \, dt = \int_0^1 (t^4 + t) \times \sqrt{3} \, dt
\]

which simplifies to:

\[
\sqrt{3} \int_0^1 (t^4 + t) \, dt
\]

Step 5: Compute the Integral

Calculate the integral:

$$
\int_0^1 t^4 \, dt = \left[\frac{t^5}{5} \right]_0^1 = \frac{1}{5}
$$

and

$$
\int_0^1 t \, dt = \left[\frac{t^2}{2} \right]_0^1 = \frac{1}{2}
$$

Therefore,

$$
\sqrt{3} \left(\frac{1}{5} + \frac{1}{2} \right)
$$

Combine the fractions:

$$
\frac{1}{5} + \frac{1}{2} = \frac{2}{10} + \frac{5}{10} = \frac{7}{10}
$$

The final value of the line integral:

$$
\boxed{
\int_{C} F \, ds = \sqrt{3} \times \frac{7}{10} = \frac{7 \sqrt{3}}{10}
}
$$

Summary and Final Result

By parameterizing the line segment from $(0, 0, 0)$ to $(1, 1, 1)$, expressing the function F in terms of the parameter, computing the differential arc length, and performing the integral, we've determined that:

$$
\boxed{
\int_{C} (x^3 y + z) \, ds = \frac{7 \sqrt{3}}{10}
}
$$

This process exemplifies how line integrals of scalar functions are evaluated in three-dimensional space, emphasizing the importance of parameterization, substitution, and integration techniques.

Applications of Line Integrals in Physics and Engineering

Understanding how to compute line integrals such as this is fundamental in various scientific fields, including:

- Physics: Calculating work done by a force field along a path.
- Engineering: Determining flux and other properties in fluid dynamics and electromagnetism.
- Mathematics: Analyzing scalar and vector fields, studying properties like potential functions.

Conclusion

Line integrals are powerful tools for analyzing functions along curves in space. The approach involves:

- Choosing an appropriate parameterization of the curve.
- Expressing the function in terms of the parameter.
- Calculating the differential arc length.
- Setting up and evaluating the integral.

Mastering these steps provides a solid foundation for tackling more complex problems in multivariable calculus and related disciplines.

Frequently Asked Questions

What is the main goal when integrating the function F = x 3y + z over a line segment in 3D space?

The main goal is to compute the line integral of the given vector or scalar function along the specified path from (0,0,0) to (1,1,1).

How do you parametrize the line segment from (0,0,0) to (1,1,1)?

A common parametrization is r(t) = (t, t, t) for t in [0, 1], since it linearly interpolates between the two points.

What is the next step after parametrizing the line segment?

Calculate the differential dr/dt and find dr to substitute into the integral, then express the function F in terms of the parameter t.

How do you evaluate the function F = x 3y + z along the line segment?

Substitute x = t, y = t, z = t into the function to get F(t) = t 3t + t = 3t^2 + t.

What is the differential element dr in terms of t for the parametrized line?

Since r(t) = (t, t, t), dr/dt = (1, 1, 1), so dr = (1, 1, 1) dt.

How do you set up the line integral once F(t) and dr are known?

The line integral is $\int_0^1$ F(t) |dr/dt| dt, but if F is scalar, it is $\int_0^1$ F(t) (dr/dt)·(unit tangent vector) dt; in this case, simplify accordingly.

What is the value of |dr/dt| for the parametrization r(t) = (t, t, t)?

Since dr/dt = (1, 1, 1), its magnitude is $\sqrt{(1^2 + 1^2 + 1^2)}$ = √3.

How do you compute the final value of the line integral for this function?

Calculate $\int_0^1$ (3t^2 + t) √3 dt, evaluate the integral term-by-term, and multiply by √3 to obtain the final result.

What is the final evaluated value of the line integral of F = x 3y + z from (0,0,0) to (1,1,1)?

The integral evaluates to (√3) [t^3 + (1/2) t^2] from 0 to 1, which equals (√3) (1 + 1/2) = (√3) (3/2) = (3√3)/2.

Additional Resources

Integrate The Function F = x^3y + z Over The Line Segment From The Point (0,0,0) To The Point (1,1,1)

Introduction

In the realm of vector calculus and multivariable integration, line integrals serve as a fundamental tool for evaluating the work done by a force field, calculating circulation, and analyzing flux along specified paths. Among the myriad of functions and paths studied, the integration of scalar functions over line segments provides foundational understanding for more complex applications in physics, engineering, and mathematics.

This article delves into the problem of integrating the scalar function \(F = x^3 y + z \) along the straight line segment connecting the points \((0, 0, 0) $ and $(1, 1, 1)$. The goal is to perform a detailed, step-by-step analysis, illustrating the methods, calculations, and underlying principles involved in such an integral.

Formulating the Problem

The problem can be summarized as:

> Compute the line integral
>
> $$
> \int_{C} (x^3 y + z) \, ds
> $$
>
> where C is the line segment from $(0, 0, 0)$ to $(1, 1, 1)$.

Key Points:

- The scalar function $F(x, y, z) = x^3 y + z$ is to be integrated along a specified path.
- The path C is a straight line segment connecting two points in 3D space.
- The line integral of a scalar function over a curve is generally expressed as:

$$
\int_{C} F \, ds
$$

Mathematical Background and Definitions

Before proceeding with the actual calculation, it is essential to understand the fundamental components involved:

Line Segment Parameterization

A line segment from point $\mathbf{r}_0 = (x_0, y_0, z_0)$ to $\mathbf{r}_1 = (x_1, y_1, z_1)$ can be parameterized as:

$$
\mathbf{r}(t) = \mathbf{r}_0 + t (\mathbf{r}_1 - \mathbf{r}_0), \quad t \in [0, 1]
$$

In our case:

$$
\mathbf{r}(t) = (0, 0, 0) + t (1, 1, 1) = (t, t, t), \quad t \in [0, 1]
$$

This parameterization simplifies the computation by expressing x, y, z in terms of a single parameter t.

Differential Arc Length ds

The differential element along the curve, ds, relates to the parameter t via:

\[
ds = |\mathbf{r}'(t)| dt
\]

where

\[
\mathbf{r}'(t) = \frac{d}{dt} \mathbf{r}(t) = (1, 1, 1)
\]

and

\[
|\mathbf{r}'(t)| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}
\]

So, \(ds = \sqrt{3} dt \).

Step-by-Step Solution

1. Substituting the Parameterization into \(F \)

Using the parameterization \(\mathbf{r}(t) = (t, t, t) \):

\[
x(t) = t, \quad y(t) = t, \quad z(t) = t
\]

The function becomes:

\[
F(t) = x^3 y + z = (t)^3 (t) + t = t^4 + t
\]

---

2. Expressing \(ds \) in terms of \(dt \)

From above:

\[
ds = |\mathbf{r}'(t)| dt = \sqrt{3} dt
\]

---

3. Setting up the integral

The line integral transforms into:

\[
\int_{t=0}^{1} F(t) |\mathbf{r}'(t)| dt = \int_{0}^{1} (t^4 + t) \sqrt{3} dt
\]

which simplifies to:

\[

$$
\sqrt{3} \int_{0}^{1} (t^4 + t) dt
$$

---

4. Computing the integral

Compute:

$$
\int_{0}^{1} t^4 dt = \left[\frac{t^5}{5} \right]_0^1 = \frac{1}{5}
$$

and

$$
\int_{0}^{1} t dt = \left[\frac{t^2}{2} \right]_0^1 = \frac{1}{2}
$$

Thus, the overall integral is:

$$
\sqrt{3} \left(\frac{1}{5} + \frac{1}{2} \right) = \sqrt{3} \left(\frac{2}{10} + \frac{5}{10} \right) = \sqrt{3} \times \frac{7}{10}
$$

Final Result

$$
\boxed{
\int_{C} (x^3 y + z) \, ds = \frac{7}{10} \sqrt{3}
}
$$

This value represents the scalar line integral of the function F along the specified line segment.

In-Depth Analysis and Implications

Geometric Interpretation

The integral calculated essentially measures the accumulated weighted "height" of the function F along the straight path from the origin to the point $(1,1,1)$. Since the function involves polynomial and linear terms, the integral captures the combined growth of x and y in the cubic term, as well as the linear increase of z.

Significance in Physical Contexts

In physics, such integrals might represent work done by a scalar potential field along a path or the total energy accumulated when traversing a specific route in a scalar field environment. For example, if F represented a potential energy density, then integrating over a path would give the total potential energy change.

Challenges and Considerations

- Path dependency: Since the function F is scalar and not necessarily conservative, the value of the integral depends solely on the path, but in this particular case, the path is a straight line, simplifying the process.
- Extension to more complex paths: For curved or piecewise paths, parameterizations become more intricate, and integrals might require numerical methods.

Broader Applications and Related Topics

This calculation exemplifies fundamental techniques in multivariable calculus, including:

- Line integrals of scalar functions
- Parameterization of curves in three dimensions
- Computing integrals involving polynomial functions

Such techniques are vital in fields like fluid dynamics, electromagnetism, and engineering design, where understanding how scalar quantities accumulate along paths is crucial.

Conclusion

The integration of the scalar function $F = x^3 y + z$ along the line segment from $(0, 0, 0)$ to $(1, 1, 1)$ illustrates core principles of multivariable calculus, combining parameterization, differential calculus, and integral computation. The result,

$$
\frac{7}{10} \sqrt{3}
$$

not only provides a numerical answer but also emphasizes the importance of methodical approaches to understanding and solving line integrals in higher dimensions.

This detailed examination underscores the importance of mastering parametrization techniques, integral calculus, and their applications across scientific and engineering disciplines. As problems grow in complexity, these foundational methods serve as essential tools for analysis and problem-solving in advanced mathematics and applied sciences.

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