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Let X(t) And X(s) Be A Laplace Transform Pair. The Laplace Transform Of X(2t) Is 0.5X(0.5s) According

Understanding the relationship between a time-domain function and its Laplace transform is fundamental in engineering, physics, and mathematics. When dealing with transformations such as $X(t)$ and $X(s)$, it is essential to grasp how modifications in the time domain affect their counterparts in the complex frequency domain. One particularly interesting scenario involves scaling the time variable and observing the corresponding effect on the Laplace transform. Specifically, if $X(t)$ and $X(s)$ form a Laplace transform pair, then examining the transform of $X(2t)$ reveals important properties related to time-scaling. This article delves into the mathematical foundation behind this relationship, explores the derivation of the Laplace transform of $X(2t)$, and discusses practical applications, including signal processing, control systems, and differential equations.

Understanding Laplace Transform Pairs

Definition of the Laplace Transform

The Laplace transform is a powerful integral transform widely used to convert functions from the time domain to the complex frequency domain. For a given function $x(t)$, defined for $t \geq 0$, the Laplace transform $X(s)$ is given by:

$$
X(s) = \mathcal{L}\{x(t)\} = \int_0^\infty e^{-st} x(t) \, dt,
$$

where s is a complex variable, $s = \sigma + j\omega$, with σ and ω representing real numbers.

Laplace Transform Pairs

A pair $(x(t), X(s))$ is called a Laplace transform pair if:

$$
X(s) = \mathcal{L}\{x(t)\},
$$

and conversely,

$$
x(t) = \mathcal{L}^{-1}\{X(s)\}.
$$

Some common Laplace transform pairs include:

- $x(t) = 1 \Rightarrow X(s) = \frac{1}{s}$,
- $x(t) = t^n \Rightarrow X(s) = \frac{n!}{s^{n+1}}$,
- $x(t) = e^{at} \Rightarrow X(s) = \frac{1}{s - a}$,
- $x(t) = \sin(\omega t) \Rightarrow X(s) = \frac{\omega}{s^2 + \omega^2}$,
- $x(t) = \cos(\omega t) \Rightarrow X(s) = \frac{s}{s^2 + \omega^2}$.

The Effect of Time Scaling on the Laplace Transform

Time Scaling Property

One of the essential properties of the Laplace transform is how it responds to scaling the time variable. If $x(t)$ and $X(s)$ form a Laplace transform pair, then the time-scaled function $x(a t)$, with $a > 0$, has a related Laplace transform:

$$
\mathcal{L}\{x(a t)\} = \frac{1}{a} X\left(\frac{s}{a}\right),
$$

where a is a positive real number representing the scale factor in the time domain.

This property indicates that compressing or stretching the original function in time corresponds to stretching or compressing its Laplace transform in the s-domain, accompanied by a scaling factor $1/a$.

Mathematical Derivation of the Property

Starting from the definition:

$$
\mathcal{L}\{x(a t)\} = \int_0^\infty e^{-s t} x(a t) \, dt.
$$

Make a substitution:

$$
u = a t \Rightarrow t = \frac{u}{a}, \quad dt = \frac{du}{a}.
$$

The limits change accordingly:

$$
t = 0 \Rightarrow u = 0, \quad t \to \infty \Rightarrow u \to \infty.
$$

Substituting into the integral:

$$
\mathcal{L}\{x(a t)\} = \int_0^\infty e^{-\frac{s u}{a}} x(u) \frac{du}{a} = \frac{1}{a} \int_0^\infty e^{-\frac{s}{a} u} x(u) \, du.
$$

Recognizing this as the Laplace transform of $x(t)$:

$$
\Rightarrow \mathcal{L}\{x(a t)\} = \frac{1}{a} X\left(\frac{s}{a}\right).
$$

Application to $X(2t)$ and the Given Transform Pair

Given Relationship

Suppose $X(t)$ and $X(s)$ form a Laplace pair. The problem statement indicates that the Laplace transform of $X(2t)$ is:

$$
\mathcal{L}\{X(2t)\} = 0.5 \, X(0.5 s).
$$

This statement aligns with the time-scaling property, but with a specific scaling factor and a coefficient, which warrants further analysis.

Deriving the Transform of $X(2t)$

Using the property outlined above, for $x(t) = X(t)$:

$$
\mathcal{L}\{X(2t)\} = \frac{1}{2} X\left(\frac{s}{2}\right).
$$

This matches the general property:

$$
\mathcal{L}\{x(a t)\} = \frac{1}{a} X\left(\frac{s}{a}\right),
$$

with $a = 2$. Therefore, in the context of $X(t)$ and $X(s)$:

$$
\boxed{
\mathcal{L}\{X(2t)\} = \frac{1}{2} X\left(\frac{s}{2}\right).
}
$$

However, the problem statement indicates that the transform of $X(2t)$ is $0.5 X(0.5 s)$, which suggests a different scaling in the argument of X, hinting at a possible typographical or interpretive nuance.

Important note: The notation $X(t)$ and $X(s)$ can sometimes be confusing. Usually, $X(t)$ denotes a time-domain function, and $X(s)$ its Laplace transform. If the problem states "Let $X(t)$ and $X(s)$ be a Laplace transform pair," then $X(t)$ is the original function, and $X(s)$ its transform.

Assuming the context is consistent with the standard notation, the key takeaway is:

$$
\mathcal{L}\{X(2t)\} = \frac{1}{2} X\left(\frac{s}{2}\right).
$$

Implications and Practical Applications

Signal Processing

Understanding how time scaling affects the Laplace transform is crucial in signal processing. For example:

- Time compression (e.g., $x(2t)$) results in a stretching of the spectrum $X(s)$,
- Time dilation impacts the frequency content and decay rates of signals.

This helps in designing filters, analyzing signals, and understanding their spectral properties.

Control Systems Engineering

In control systems, the response of systems to scaled inputs is often analyzed using the Laplace transform. Recognizing how the Laplace domain transforms under time scaling enables engineers to:

- Predict system response to scaled inputs,
- Simplify complex differential equations,
- Design controllers that accommodate time-scaling effects.

Differential Equations

Transforming differential equations often involves understanding how functions behave under scaling. For instance, solutions involving $x(2t)$ can be derived by applying the time-scaling property, simplifying the solving process.

Summary and Key Takeaways

- The Laplace transform of a scaled function $x(a t)$ is given by:

$$
\mathcal{L}\{x(a t)\} = \frac{1}{a} X\left(\frac{s}{a}\right),
$$

where $X(s) = \mathcal{L}\{x(t)\}$.

- When $a = 2$, the transform becomes:

$$
\mathcal{L}\{x(2 t)\} = \frac{1}{2} X\left(\frac{s}{2}\right).
$$

- The relationship between the time domain and the s-domain under scaling is inverse: scaling time by a factor compresses or stretches the spectrum in the s-domain accordingly.

- Recognizing

Frequently Asked Questions

What does the given transformation tell us about the relationship between X(t) and X(s)?

It indicates how the Laplace transform of a scaled time function, specifically X(2t), relates to the original transform X(s), showing a scaling property in the Laplace domain.

Why is the Laplace transform of X(2t) equal to 0.5 times X(0.5s)?

This is due to the time-scaling property of Laplace transforms, which states that scaling the time variable by a factor 'a' results in the Laplace transform being scaled and evaluated at a scaled argument, along with a factor of 1/a.

What is the general formula for the Laplace transform of a time-scaled function X(at)?

The general formula is L{X(at)} = (1/|a|) X(s/a), where a is a non-zero real number.

In the context of the given transformation, what is the value of 'a'?

The value of 'a' is 2, since the transformation involves X(2t).

How does the scaling property affect the shape of the original function X(t)?

Scaling the time variable compresses or stretches the original function along the time axis, which in the Laplace domain results in a corresponding evaluation at a scaled s with a magnitude adjustment.

Is the factor 0.5 in the transformed Laplace domain consistent with the standard scaling property?

Yes, because for a scaling factor 'a' = 2, the Laplace transform includes a factor of 1/|a| = 0.5, matching the given 0.5X(0.5s).

Can this property be used to find the inverse Laplace transform of scaled functions?

Yes, understanding the scaling property helps in deriving the inverse Laplace transform of scaled functions by relating them back to the original function.

What are practical applications of the time-scaling property in engineering?

It is used in signal processing, control systems, and system analysis to analyze how signals behave under time compression or expansion, enabling easier manipulation and understanding of system responses.

Additional Resources

Laplace Transform of X(2t): An In-Depth Analysis

Understanding the behavior of Laplace transforms under various time-scaling operations is fundamental in control theory, signal processing, and differential equations. The statement, "Let X(t) and X(s) be a Laplace transform pair. The Laplace transform of X(2t) is 0.5X(0.5s) because..." encapsulates a key property related to how the Laplace transform responds to scaling of the time domain. This principle, often referred to as the time-scaling property, reveals the elegant interplay between the time domain and the complex frequency domain. In this comprehensive review, we will explore the mathematical foundations, practical implications, and applications of this property, ensuring a clear understanding of its significance in both theoretical and applied contexts.

Fundamentals of the Laplace Transform and Time-Scaling Property

What is the Laplace Transform?

The Laplace transform is a powerful integral transform used to convert functions from the time domain into the complex frequency domain. It is defined as:

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^{\infty} x(t) e^{-st} dt$$

where:
- $x(t)$ is a real-valued function representing a signal or system response.
- s is a complex number $s = \sigma + j\omega$, with σ and ω real numbers.

This transform simplifies the analysis of linear time-invariant (LTI) systems, especially differential equations, by converting them into algebraic equations in the s-domain.

The Time-Scaling Property

One of the key properties of the Laplace transform is its response to scaling in the time domain:

$$\mathcal{L}\{x(at)\} = \frac{1}{|a|} X \left(\frac{s}{a} \right)$$

where a is a non-zero real number.

Applying this property to the specific case where $a=2$, the transform becomes:

$$\mathcal{L}\{x(2t)\} = \frac{1}{2} X \left(\frac{s}{2} \right)$$

This formula indicates that compressing a signal in time (by a factor of 2) stretches its Laplace transform along the s-axis and scales it by $1/2$.

Derivation of the Laplace Transform of $X(2t)$

Step-by-step Derivation

Let's derive the Laplace transform of $x(2t)$:

$$
\begin{aligned}
\mathcal{L}\{x(2t)\} &= \int_0^{\infty} x(2t) e^{-st} dt
\end{aligned}
$$

Make a substitution to simplify the integral:

Let $u = 2t \Rightarrow t = \frac{u}{2}$

Then, $dt = \frac{du}{2}$, and when $t \to \infty$, $u \to \infty$; when $t=0$, $u=0$.

Replacing in the integral:

$$
\begin{aligned}
\mathcal{L}\{x(2t)\} &= \int_0^{\infty} x(u) e^{-s \frac{u}{2}} \frac{du}{2} \\
&= \frac{1}{2} \int_0^{\infty} x(u) e^{-\frac{s}{2} u} du \\
&= \frac{1}{2} X \left(\frac{s}{2} \right)
\end{aligned}
$$

Thus, the Laplace transform of $x(2t)$ is:

$$
\boxed{
\mathcal{L}\{x(2t)\} = \frac{1}{2} X \left(\frac{s}{2} \right)
}
$$

which aligns with the general time-scaling property.

Implications and Applications of the Scaling Property

Signal Analysis and System Response

- Time Compression and Expansion: The property illustrates how a signal's time-scale modification affects its Laplace transform, enabling engineers to analyze systems responding to signals that are compressed or stretched in time.
- Design of Filters: By understanding how scaling transforms the system's transfer function, designers can craft filters that compensate for or leverage these effects.

Control Systems

- Stability Analysis: The shift in the s-domain caused by time-scaling can influence the pole locations, affecting system stability.
- Inverse Transform Applications: Recognizing the scaled relationship simplifies inverse Laplace transform calculations when dealing with scaled inputs.

Practical Examples

- Signal Time-Scaling: In audio processing, a sound wave compressed in time (fast playback) corresponds to a scaled Laplace transform, aiding in analysis.
- Mechanical Vibrations: Rescaling time in vibration signals allows for the assessment of system behavior under different temporal conditions.

Features, Pros, and Cons of the Time-Scaling Property

Features

- Directly relates time-scaling in the physical domain to a predictable transformation in the s-domain.
- Simplifies the analysis of signals undergoing compression or expansion.
- Facilitates inverse transform calculations for scaled signals.

Pros

- Mathematical Elegance: Provides a straightforward way to analyze complex time-scaling effects.
- Versatility: Applicable across engineering disciplines involving differential equations and system analysis.
- Computational Efficiency: Reduces complex integral calculations to simpler algebraic manipulations.

Cons

- Limited to Non-zero Scaling Factors: The property is valid only for $a \neq 0$.
- Assumption of Causality: Typically assumes signals are causal, which may not always hold in practical scenarios.
- Potential for Confusion: Misapplication or misinterpretation can lead to errors, especially in inverse transformations.

Limitations and Considerations

While the time-scaling property is powerful, several considerations must be kept in mind:

- Causality and Domain Restrictions: The derivation assumes causal signals defined for $t \geq 0$. For non-causal signals, modifications are necessary.
- Inverse Transform Complexity: While forward transforms are straightforward, inverse transforms of scaled functions can sometimes be non-trivial.
- Numerical Computation: Real-world signals may require numerical methods for Laplace transforms, which can introduce approximation errors.

Extensions and Related Properties

Beyond the basic scaling property, the Laplace transform exhibits several related features:

- Time-Shift Property: $\mathcal{L}\{x(t - t_0)u(t - t_0)\} = e^{-st_0} X(s)$
- Frequency Shift: $\mathcal{L}\{e^{at} x(t)\} = X(s - a)$
- Differentiation in Time Domain: Corresponds to multiplication by s in the s-domain, aiding in solving differential equations.

Understanding these properties enhances the analyst’s toolkit for system analysis and signal processing.

Conclusion

The statement, "The Laplace Transform of \(X(2t) \) is \(0.5 X(0.5s) \)", encapsulates the fundamental time-scaling property of the Laplace transform. This property not only provides a direct mathematical relationship between scaled signals and their transforms but also offers practical insights into how signals behave under compression or expansion in time. Its applications span numerous fields, including control systems, electrical engineering, mechanical vibrations, and signal processing. By mastering this property, engineers and mathematicians can simplify complex analyses, design more robust systems, and develop a deeper understanding of the interplay between the time and frequency domains. Though it has limitations and requires careful application, its elegance and utility make it an essential concept in the toolbox of anyone working with linear systems and transforms.

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