Nd The First Three Nonzero Terms In The Power Series Expansion For The Product F(x)g(x) Where F(x)=ex

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Understanding power series expansions is fundamental in many areas of calculus, differential equations, and mathematical analysis. In this article, we explore the process of determining the first three nonzero terms in the power series expansion for the product $F(x)g(x)$, where $F(x) = e^x$. This topic is essential for students and professionals working with series solutions, approximations, and functional analysis, offering both theoretical insights and practical applications.

Introduction to Power Series and Their Significance

Power series are infinite sums of the form:

$$
\sum_{n=0}^\infty a_n (x - c)^n
$$

where a_n are coefficients, and c is the center of the expansion. They serve as powerful tools for representing functions locally around a point, especially when functions are difficult to evaluate directly.

Why Use Power Series?

- Simplify complex functions into polynomial approximations.
- Facilitate analysis of function behavior near specific points.
- Enable solving differential equations via series solutions.
- Approximate functions numerically with controlled accuracy.

Key Concepts

- Radius of convergence: The interval within which the series converges to the function.
- Term-by-term operations: Addition, subtraction, multiplication, and differentiation can often be performed directly on series.

Understanding the Function $F(x) = e^x$

The exponential function e^x is one of the most important functions in mathematics, with a well-known power series expansion centered at 0:

$$
e^x = \sum_{n=0}^\infty \frac{x^n}{n!}
$$

Series Expansion of e^x:

$$
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots
$$

This series converges for all real x, making it an entire function.

Given Function $g(x)$: Assumptions and General Form

To analyze the product $F(x)g(x) = e^x g(x)$, we need to understand or specify the form of $g(x)$. Since the user hasn't provided a specific $g(x)$, the most common approach is to express $g(x)$ as a power series:

$$
g(x) = \sum_{n=0}^\infty b_n x^n
$$

where b_n are the coefficients of $g(x)$. This general form allows us to derive the power series expansion of the product $e^x g(x)$.

Power Series Expansion of the Product $F(x)g(x)$

The product of two functions expressed as power series can be obtained by the Cauchy product rule. If:

$$
F(x) = \sum_{n=0}^\infty a_n x^n
$$

and

$$
g(x) = \sum_{n=0}^\infty b_n x^n
$$

then their product:

$$
F(x)g(x) = \sum_{n=0}^\infty c_n x^n
$$

where the coefficients c_n are given by:

$$
c_n = \sum_{k=0}^n a_k b_{n-k}
$$

This convolution sum allows us to compute the power series expansion of the product term-by-term.

Calculating the First Three Nonzero Terms of $F(x)g(x)$

To find the first three nonzero terms, we'll consider the series expansions for both $F(x) = e^x$ and $g(x)$, then apply the Cauchy product to derive the initial terms.

Step 1: Series for e^x

$$
e^x = \sum_{n=0}^\infty \frac{x^n}{n!}
$$

The first few terms are:

$$
1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots
$$

Step 2: Series for $g(x)$

Assuming a general power series:

$$
g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots
$$

The coefficients $b_0, b_1, b_2, \ldots$ depend on the specific function $g(x)$, but for a general analysis, we keep them symbolic.

Step 3: Compute the coefficients c_n

Using $c_n = \sum_{k=0}^n a_k b_{n-k}$, where $a_k = \frac{1}{k!}$, the first few coefficients are:

- Coefficient c_0:

$$
c_0 = a_0 b_0 = 1 \times b_0 = b_0
$$

- Coefficient c_1:

$$
c_1 = a_0 b_1 + a_1 b_0 = 1 \times b_1 + 1 \times b_0 = b_0 + b_1
$$

- Coefficient c_2:

$$
c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0 = 1 \times b_2 + 1 \times b_1 + \frac{1}{2} b_0
$$

- Coefficient c_3:

$$
c_3 = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 = 1 \times b_3 + 1 \times b_2 + \frac{1}{2} b_1 + \frac{1}{6} b_0
$$

Step 4: Write the partial series expansion

$$
F(x)g(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots
$$

Substituting the coefficients, the first few terms are:

$$
b_0 + (b_0 + b_1) x + \left(b_2 + b_1 + \frac{b_0}{2} \right) x^2 + \left(b_3 + b_2 + \frac{b_1}{2} + \frac{b_0}{6} \right) x^3 + \ldots
$$

Step 5: Identify the first three nonzero terms

Depending on the specific $g(x)$, the initial nonzero terms vary. For example:

- If $g(0) \neq 0$ (i.e., $b_0 \neq 0$), then the first term is b_0.
- The second term involves $b_0 + b_1$.
- The third involves $b_2 + b_1 + \frac{b_0}{2}$.

Example:

Suppose $g(x) = 1 + x + x^2$, so:

$$
b_0 = 1, \quad b_1 = 1, \quad b_2 = 1
$$

Then, the first three nonzero terms are:

1. $c_0 = 1$
2. $c_1 = 1 + 1 = 2$
3. $c_2 = 1 + 1 + \frac{1}{2} = 2.5$

and the expansion begins as:

$$
F(x)g(x) \approx 1 + 2x + 2.5 x^2
$$

Applications of the Power Series Expansion in Various Fields

Understanding the initial terms of the power series for $F(x)g(x)$ has numerous applications across mathematics and engineering.

1. Approximate Solutions in Differential Equations

Series solutions are vital for solving differential equations where closed-form solutions are difficult or impossible to find. By truncating the series after the first few terms, approximate solutions can be obtained near a point of expansion.

2. Numerical Analysis and Computations

Power series provide a basis for numerical algorithms, enabling the approximation of functions with high accuracy within the radius of convergence. For example, computing $e^x g(x)$ for small x can be efficiently performed using the initial terms.

3. Signal Processing and Engineering

In signal analysis, series expansions help in understanding the frequency components and behavior of complex signals, especially when dealing with convolutions that relate directly to power series products.

4. Theoretical Insights in Mathematics

Series expansions reveal properties like analyticity, singularities, and asymptotic behavior of functions, aiding in proofs and theoretical exploration.

Summary and Key Takeaways

- Power series expansions are powerful tools for representing and approximating functions around specific points.
- The product of two functions expressed as power series can be computed using the C

Frequently Asked Questions

What are the first three nonzero terms in the power series expansion for F(x) = e^x?

The first three nonzero terms are 1, x, and x^2/2.

How do you find the power series expansion for the product F(x)g(x) where F(x) = e^x?

You multiply the power series of e^x by the power series of g(x), then combine like terms to find the first nonzero terms.

What is the significance of the first three nonzero terms in the product expansion for F(x)g(x)?

They provide the initial approximation of the product near x = 0, useful in analysis and solving differential equations.

If F(x) = e^x, how does its power series influence the initial terms of the product with another function g(x)?

Since e^x expands to 1 + x + x^2/2 + ..., it multiplies each term of g(x)'s series, affecting the first few terms of the product accordingly.

Can the first three nonzero terms of F(x)g(x) be used to approximate the product near zero? How?

Yes, by truncating the series to these terms, you can approximate F(x)g(x) near x = 0, which is useful for small x analyses.

Additional Resources

Nd The First Three Nonzero Terms In The Power Series Expansion For The Product F(x)g(x) Where F(x)=e^x

Understanding the power series expansions of functions plays a vital role in mathematical analysis, especially in areas like calculus, differential equations, and mathematical modeling. A fundamental concept in this realm is the expansion of functions into infinite series, which allows us to approximate complex functions with polynomials that are easier to manipulate and analyze. In particular, exploring the power series expansion of the product of two functions, especially when one of them is a well-understood exponential function, reveals insightful properties and simplifies calculations significantly. This article delves into the specifics of finding the first three nonzero terms in the power series expansion for the product $F(x)g(x)$, where $F(x) = e^x$, emphasizing the process, significance, and applications of such an expansion.

Understanding Power Series and Their Significance

Before diving into the specifics of the product expansion, it is essential to grasp what power series are and why they matter.

What Is a Power Series?

A power series is an infinite sum of the form:

$$
\sum_{n=0}^{\infty} a_n (x - c)^n
$$

where:
- a_n are the coefficients,
- c is the center of the expansion (often zero),

- x is the variable.

When $c = 0$, the series is called a Maclaurin series; when $c \neq 0$, it's a Taylor series centered at c.

Power series allow us to approximate functions locally around a point c, and within their radius of convergence, these series can be used to evaluate functions, solve differential equations, and analyze function behavior.

Why Focus on the First Few Nonzero Terms?

In practical applications, infinite series are approximated by their first few terms. The first nonzero terms often give a good approximation near the expansion point, especially when the higher-order terms are negligible within a certain domain. Identifying these terms helps in:
- Simplifying complex calculations,
- Understanding the local behavior of functions,
- Developing intuition about the influence of different terms.

Power Series Expansion of $F(x) = e^x$

The exponential function e^x is one of the most fundamental functions in mathematics, with a well-known power series expansion:

$$
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}
$$

This series converges for all real x, making e^x an entire function.

First Three Nonzero Terms of e^x

The first three terms are straightforward:

$$
e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots
$$

The first three nonzero terms are:

1. 1 (constant term)
2. x (linear term)
3. $\frac{x^2}{2}$ (quadratic term)

These terms are essential for approximating e^x near $x = 0$.

Power Series Expansion of $g(x)$

The nature of $g(x)$ determines how its power series expansion looks and how it interacts with the exponential function. Let's assume $g(x)$ has a known power series expansion:

$$
g(x) = \sum_{n=0}^{\infty} b_n x^n
$$

where b_n are the coefficients that depend on $g(x)$.

For example, consider a polynomial $g(x)$ with known initial terms:

$$
g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots
$$

In this case, the first three non-zero terms of $g(x)$ are:

- a_0,
- $a_1 x$,
- $a_2 x^2$.

Finding the Power Series Expansion of the Product $F(x)g(x) = e^x \cdot g(x)$

The core of this discussion revolves around deriving the first three nonzero terms of the product $F(x)g(x)$. Since $F(x) = e^x$, and $g(x)$ has a known power series, the product can be expanded using the Cauchy product of series.

The Cauchy Product Formula

Given:

$$
F(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}
$$

and

$$
g(x) = \sum_{n=0}^{\infty} b_n x^n,
$$

the product $F(x)g(x)$ is:

$$
F(x)g(x) = \sum_{n=0}^{\infty} c_n x^n,
$$

where

$$
c_n = \sum_{k=0}^n \frac{b_k}{(n - k)!}
$$

This formula allows us to compute the coefficients c_n by summing products of the coefficients of e^x and $g(x)$.

Calculating the First Three Nonzero Terms

To find the first three nonzero terms, we need to determine c_0, c_1, c_2.

1. Coefficient c_0:

$$
c_0 = b_0 \times \frac{1}{0!} = b_0 \times 1 = b_0
$$

2. Coefficient c_1:

$$
c_1 = b_0 \times \frac{1}{1!} + b_1 \times \frac{1}{0!} = b_0 \times 1 + b_1 \times 1 = b_0 + b_1
$$

3. Coefficient c_2:

$$
c_2 = b_0 \times \frac{1}{2!} + b_1 \times \frac{1}{1!} + b_2 \times \frac{1}{0!} = b_0 \times \frac{1}{2} + b_1 \times 1 + b_2 \times 1
$$

Thus,

$$
c_2 = \frac{b_0}{2} + b_1 + b_2
$$

The first three nonzero terms of $F(x)g(x)$ are then:

$$
c_0 + c_1 x + c_2 x^2
$$

which explicitly depend on the coefficients of $g(x)$.

Application Examples

Let's consider specific examples to illustrate how this process works in practice.

Example 1: $g(x) = a_0 + a_1 x + a_2 x^2$

Suppose $g(x)$ is a quadratic polynomial:

$$
g(x) = a_0 + a_1 x + a_2 x^2
$$

then the first three coefficients:

- $b_0 = a_0$,
- $b_1 = a_1$,
- $b_2 = a_2$.

The first three terms of the product are:

$$
\boxed{
\begin{aligned}
& c_0 = a_0 \\
& c_1 = a_0 + a_1 \\
& c_2 = \frac{a_0}{2} + a_1 + a_2
\end{aligned}
}
$$

The approximate expansion:

$$
F(x)g(x) \approx a_0 + (a_0 + a_1) x + \left(\frac{a_0}{2} + a_1 + a_2\right) x^2
$$

This form is very useful for approximating the product near $x = 0$.

Example 2: $g(x) = e^x$ itself

If $g(x) = e^x$, then the series expansion of $g(x)$ is:

$$
g(x) = \sum_{n=0}^\infty \frac{x^n}{n!}
$$

which means:

$$
b_n = \frac{1}{n!}
$$

Calculating the first three coefficients:

- $c_0 = b_0 = 1$,
- $c_1 = b_0 + b_1 = 1 + 1 = 2$,
- $c_2 = \frac{b_0}{2} + b_1 + b_2 = \frac{1}{2} + 1 + \frac{1}{2} = 2$.

Therefore, the first three terms of $e^x \cdot e^x = e^{2x}$ are:

$$
2 + 2x + 2x^2
$$

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