Point) Consider The Initial Value Problem2ty'= 4y, Y(-2) = -4.a. Find The Value Of The Constant C And

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Introduction to Initial Value Problems and Differential Equations

Understanding initial value problems (IVPs) is fundamental in differential equations, a vital branch of mathematics with applications across physics, engineering, biology, and economics. An IVP involves determining a function that satisfies a differential equation along with specified initial conditions. In this context, the differential equation 2ty' = 4y, with the initial condition Y(-2) = -4, presents an intriguing problem for students and professionals alike, aiming to find the particular solution and the constant of integration, often denoted as C.

This article provides a comprehensive, step-by-step approach to solving this initial value problem, emphasizing the process of solving first-order differential equations, applying initial conditions, and interpreting the solution. Whether you are a student preparing for exams or a professional seeking clarity, this guide aims to clarify the method and reasoning involved.

Understanding the Differential Equation 2ty' = 4y

The Nature of the Differential Equation

The given differential equation:

$$ 2t y' = 4 y $$

is a first-order linear differential equation. It relates the derivative of the function $y(t)$, denoted as y', to the function itself, scaled by the independent variable t.

Recognizing the Type of Differential Equation

Rearranged, the equation becomes:

$$ y' = \frac{4 y}{2 t} = \frac{2 y}{t} $$

which is a separable differential equation because the variables y and t can be separated on different sides of the equation.

Step-by-Step Solution Approach

Step 1: Rewrite the Differential Equation in Separable Form

Starting with:

$$ 2 t y' = 4 y $$

divide both sides by $2 t$:

$$ y' = \frac{4 y}{2 t} = \frac{2 y}{t} $$

Expressed as:

$$ \frac{dy}{dt} = \frac{2 y}{t} $$

This form indicates that the variables are separable:

$$ \frac{dy}{y} = \frac{2}{t} dt $$

Step 2: Separate Variables

Separate the variables:

$$ \frac{dy}{y} = \frac{2}{t} dt $$

This means all y-terms are on one side and all t-terms on the other.

Step 3: Integrate Both Sides

Integrate both sides:

$$ \int \frac{1}{y} dy = \int \frac{2}{t} dt $$

which yields:

$$ \ln |y| = 2 \ln |t| + C $$

where C is the constant of integration.

Step 4: Simplify the Solution

Recall that:

$$ \ln |y| = \ln |t|^{2} + C $$

which implies:

$$ |y| = e^{C} |t|^{2} $$

Let $K = e^{C}$, a positive constant, then:

$$y(t) = \pm K t^{2}$$

Since the constant K can be positive or negative, the general solution becomes:

$$y(t) = C t^{2}$$

where C is an arbitrary real constant.

Applying Initial Conditions to Find the Specific Constant C

Step 1: Use the Initial Condition $Y(-2) = -4$

Plug in $t = -2$ and $y = -4$ into the general solution:

$$-4 = C \times (-2)^{2}$$

Calculate $(-2)^{2}$:

$$(-2)^{2} = 4$$

So:

$$-4 = C \times 4$$

Step 2: Solve for C

Divide both sides by 4:

$$C = \frac{-4}{4} = -1$$

Result:

The particular solution to the differential equation satisfying the initial condition is:

$$y(t) = - t^{2}$$

Interpretation of the Solution

The function $y(t) = - t^{2}$ describes the specific behavior of the solution within the domain of the initial condition. This quadratic function opens downward because of the negative coefficient, and it passes through the point $(-2, -4)$ as required.

Key observations:

- The solution is valid for all $t \neq 0$ because the original differential equation involves

division by t.
- The initial condition specifies the value at $t = -2$, providing a starting point for the solution.
- The constant $C = -1$ uniquely characterizes this particular solution.

Additional Insights on the Differential Equation and Its Solution

Domain of the Solution

Since the original differential equation involves t in the denominator, the solution is valid on intervals excluding $t = 0$. Typically, the domain is:

$$t \in (-\infty, 0) \cup (0, \infty)$$

depending on initial conditions and context.

Behavior of the Solution

- For $t > 0$, the solution $y(t) = - t^{2}$ is negative and decreasing as t increases.
- For $t < 0$, the solution remains negative, matching the initial condition at $t = -2$.

Significance of the Constant C

The constant C determines the particular solution that satisfies the initial condition. Different initial conditions would lead to different values of C, resulting in a family of solutions.

Summary of Key Steps in Solving the Initial Value Problem

Step	Description	Result
1	Rewrite the differential equation in separable form	$y' = \frac{2 y}{t}$
2	Separate variables	$\frac{dy}{y} = \frac{2}{t} dt$
3	Integrate both sides	$\ln \|y\| = 2 \ln \|t\| + C$
4	Simplify to exponential form	$y(t) = C t^{2}$
5	Apply initial condition $y(-2) = -4$	$C = -1$
6	Write the particular solution	$y(t) = - t^{2}$

Practical Applications of Such Differential Equations

Differential equations like $2 t y' = 4 y$ appear in various real-world contexts:

- Physics: Modeling systems where the rate of change depends on the current state and a variable parameter.

- Biology: Population dynamics where growth rate depends on both current population and environmental factors.
- Economics: Financial models where growth depends on current value and external variables.
- Engineering: Heat transfer, signal processing, or control systems involving variable coefficients.

Understanding how to solve these equations and apply initial conditions is essential for accurately modeling and analyzing such phenomena.

Conclusion

The initial value problem involving the differential equation $2 t y' = 4 y$ with $Y(-2) = -4$ demonstrates the importance of the separation of variables technique, the application of initial conditions, and the interpretation of solutions within their domain. By carefully following the steps—rewriting, separating, integrating, and applying initial conditions—we identify the specific solution:

$$\boxed{ y(t) = - t^{2} }$$

This solution is valid and satisfies all the given conditions, exemplifying the process of solving first-order differential equations and understanding their implications in various scientific and engineering fields.

Frequently Asked Questions (FAQs)

Q1: Why is the differential equation separable?

A: Because it can be rewritten as $y' = \frac{2 y}{t}$, where y and t are on separate sides, allowing direct integration.

Q2: Can this method be applied to all differential equations?

A: No, the method shown applies to separable equations. Not all differential equations are separable; some require different techniques like integrating factors or substitution.

Q3: What happens if the initial condition is at $t = 0$?

A: Since the original differential equation involves division by t, the point $t=0$ is typically excluded from the domain, and initial conditions at $t=0$ may lead to undefined or singular solutions.

Q4: How does the constant C relate to initial conditions?

A: The constant C is determined by substituting the initial condition into the general solution, ensuring the particular solution satisfies the given point.

By mastering the process outlined above, you will be well-equipped to solve similar initial value problems involving first-order differential equations.

Frequently Asked Questions

What is the initial value problem given in the differential equation?

The initial value problem is 2ty' = 4y with y(-2) = -4.

How do you separate variables to solve the differential equation 2ty' = 4y?

Rewrite as y'/y = 2/ t and then integrate both sides to find y in terms of t.

What is the general solution to the differential equation 2ty' = 4y?

The general solution is y(t) = C t^2, where C is an arbitrary constant.

How do you determine the constant C using the initial condition y(-2) = -4?

Substitute t = -2 and y = -4 into the general solution y(t) = C t^2 to solve for C.

What is the value of the constant C for the given initial condition?

C = -4 / (-2)^2 = -4 / 4 = -1.

What is the particular solution to the initial value problem?

The particular solution is y(t) = -1 t^2 = -t^2.

Why is it important to verify the solution satisfies both the differential equation and initial condition?

Verifying ensures the solution is correct and consistent with the problem's requirements.

Additional Resources

Initial Value Problem Analysis: Solving 2ty' = 4y, with Y(-2) = -4

Introduction to Differential Equations and Initial Value Problems

Differential equations form the backbone of mathematical modeling in numerous scientific and engineering disciplines, describing how a quantity evolves over time or space under various influences. An initial value problem (IVP) specifies a differential equation together with initial conditions, allowing us to find a particular solution relevant to a specific scenario.

In this review, we focus on the specific problem:

$$ 2t y' = 4 y, \quad y(-2) = -4 $$

Our goal is to analyze this IVP comprehensively: to find the general solution, determine the constant C, and interpret the solution's behavior.

Understanding the Differential Equation: 2ty' = 4y

1. Nature of the Equation

The given differential equation:

$$ 2 t y' = 4 y $$

can be rewritten as:

$$ y' = \frac{4 y}{2 t} = \frac{2 y}{t} $$

assuming $t \neq 0$.

This is a first-order linear differential equation that is also separable, which simplifies the process of finding a solution.

2. Domain of the Solution

Since the equation involves division by t, the solution is valid for all $t \neq 0$. The initial condition is specified at $t = -2$, which is within the domain $t < 0$, so the solution around this point is valid.

Step-by-Step Solution Process

1. Recognize the Equation Type

Given the form:

$$y' = \frac{2 y}{t}$$

this is a separable differential equation, which means it can be written as:

$$\frac{dy}{dt} = \frac{2 y}{t}$$

or

$$\frac{dy}{y} = \frac{2}{t} dt$$

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2. Separate Variables

Separating variables:

$$\frac{dy}{y} = \frac{2}{t} dt$$

This step is fundamental in solving first-order differential equations: bringing all y-terms to one side and all t-terms to the other.

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3. Integrate Both Sides

Integrate both sides:

$$\int \frac{1}{y} dy = \int \frac{2}{t} dt$$

which yields:

$$ \ln | y | = 2 \ln | t | + C $$

where C is the constant of integration.

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4. Solve for y

Exponentiating both sides:

$$ | y | = e^{2 \ln | t | + C} = e^{C} e^{2 \ln | t |} $$

Recall that:

$$ e^{2 \ln | t |} = (e^{\ln | t |})^{2} = | t |^{2} $$

Let $K = e^{C}$, which is an arbitrary positive constant, capturing the entire constant of integration:

$$ | y | = K | t |^{2} $$

Thus, the general solution:

$$ y(t) = \pm K t^{2} $$

or simply:

$$ y(t) = C t^{2} $$

where C is an arbitrary real constant ($C \in \mathbb{R}$), since the sign can be incorporated into the constant C.

Determining the Constant C from the Initial Condition

The initial condition provided is:

$$ y(-2) = -4 $$

Substitute $t = -2$:

$$ y(-2) = C (-2)^{2} = C \times 4 $$

Given:

$$ y(-2) = -4 $$

we have:

$$ C \times 4 = -4 $$

which gives:

$$ C = -1 $$

Therefore, the particular solution satisfying the initial condition is:

$$ y(t) = - t^{2} $$

Interpreting the Solution and Its Behavior

1. The Explicit Solution

The unique solution to the IVP is:

$$ y(t) = - t^{2} $$

This quadratic function is valid for all $t \neq 0$, aligning with our earlier domain considerations.

2. Graphical Representation

- The solution $y(t) = - t^{2}$ describes a downward-opening parabola.
- The vertex is at the origin $(0, 0)$, but note that the initial condition is specified at $t = -2$.
- At $t = -2$, $y = -4$, matching the initial condition.
- As $t \to -\infty$, $y(t) \to - \infty$; as $t \to 0^{-}$, $y(t) \to 0^{-}$.

3. Domain and Continuity

- The solution is valid for $t \in (-\infty, 0)$ and $t \in (0, \infty)$, except at $t = 0$, where the original differential equation is undefined due to division by zero.
- The behavior near $t=0$ indicates a potential singularity or discontinuity; hence, solutions are considered on intervals excluding zero.

4. Physical Interpretation

- If modeled physically, such as in population dynamics or heat transfer, the negative quadratic might indicate a decreasing quantity over time, with the initial state specified at $t = -2$.
- The solution's sign and shape depend on the context, but mathematically, it is well-defined within its domain.

Summary of Key Findings

- The differential equation $2 t y' = 4 y$ is separable and solvable through straightforward integration.
- The general solution is $y(t) = C t^{2}$.
- The initial condition $y(-2) = -4$ yields $C = -1$.
- The particular solution is $y(t) = - t^{2}$.
- The solution is valid for all $t \neq 0$, with the understanding that $t=0$ is a singular point where the original differential equation is undefined.
- Graphically, the solution represents a downward-opening parabola crossing the point $(-2, -4)$, with the corresponding behavior as t varies.

Broader Implications and Additional Insights

1. Significance of the Constant C

The constant C encapsulates initial conditions and determines the specific trajectory of the solution within the family of solutions. Its calculation directly from initial conditions ensures that the solution aligns with real-world data or specific boundary conditions.

2. Behavior Near Singularities

The point $t=0$ is a singularity for the differential equation. While solutions are valid on intervals excluding zero, understanding the behavior as $t \to 0$ can be critical in applications. For the particular solution:

$$y(t) = - t^{2}$$

as $t \to 0$, $y(t) \to 0$, indicating the solution smoothly approaches zero.

3. Extension to Broader Contexts

- If the initial condition were given at a different point, the constant C would change accordingly.
- For more complex differential equations, such explicit solutions may not exist, requiring numerical methods or approximation techniques.
- The method demonstrated here—separation of variables—is a fundamental approach for solving many first-order differential equations.

Conclusion

Solving the initial value problem $2 t y' = 4 y$, $y(-2) = -4$, involves recognizing the separable nature of the differential equation, integrating both sides, and applying the initial condition to determine the specific constant C.

The key steps include:

- Rewriting the differential equation in a separable form.
- Performing integration to find the general solution.
- Using the initial condition to find the particular solution.

The final explicit solution:

$$y(t) = - t^{2}$$

provides a clear and interpretable function that satisfies both the differential equation and the initial condition, with domain considerations emphasizing the importance of understanding singularities and the solution's applicability.

This comprehensive analysis underscores the importance of methodical problem-solving in differential equations, illustrating how initial conditions shape solutions and how the mathematical structure guides us to the correct, unique solution relevant to the problem's context.

Note: Always consider the domain restrictions implied by the differential equation and initial conditions. For the problem at hand, solutions are valid for $t \neq 0$, with particular attention to the initial point at $t = -2$.

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